AN INTRODUCTION TO LABORATORY PHYSICS

TUTTLE

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LABORATORY PHYSICS

BY

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PREFACE

THIS book is essentially a revision of the mimeographed direction sheets that have been used in the first part of the laboratory course given by the writer at Jefferson Medical College. It does not cover the ground of the usual laboratory manuals of physics, but is intended to precede the use of any one of them in a course of physical measurement, and includes the matter that is usually relegated to an introductory chapter or an appendix, where the student is not apt to get as good a grasp of it as he does of the subjects that are emphasized by his experimental work. In addition to the statements of facts and theory each of the fifteen lessons in the book includes directions for actual experimental work to be performed by the student, and the amount of this work has been so planned that each lesson will require the same length of time as any of the others. For the average student this means about three hours, but the material of the lessons can easily be divided into a greater number of shorter exercises if desirable.

Explanations and directions have been given with considerable detail, partly in order to avoid the necessity for continuous oral assistance on the part of the instructor, and partly to help the student to learn with a minimum of deliberate memorizing. For the latter purpose facts have sometimes been stated

implicitly instead of explicitly, and later have been reiterated in a more expositional form.

At first glance the book will seem more mathematical than it really is, for the re-statement of some elementary principle is occasionally helpful to any student. No knowledge of trigonometry, however, is presupposed, and none is imposed upon the reader of the book, the terms "function," "tangent," "cosine," etc., that will occasionally be found being used merely as convenient abbreviations for ideas that would otherwise need a more cumbrous description.

In the introductory chapter the commonest mathematical deficiencies of the student are reviewed and an opportunity is given him to test his weak points. A lesson on logarithms is included, which can be omitted, if preferred, by a class that is familiar with the subject; but there are often members of such a class who cannot make practical use of logarithmic tables readily, or even accurately, without additional practice, and to anyone who does not need the practice it will not be at all irksome. Care has been taken to make the tables both accurate and convenient. Experience has shown that the somewhat unconventional arrangement of the table of probable errors (page 141) is the most satisfactory in actual use. The table of logarithmic circular functions has been given the greatest possible compactness. The columns of the table of four-place logarithms are arranged especially for the convenience of the student of physics and the proportional parts are given in the same way as in carefully constructed larger tables. None of the methods of arranging a five-place table with proportional parts

within the limits of two pages has ever succeeded in giving the fifth figure satisfactorily, and several books for physicists have been published in which even the fourth figure of such a table will often be found incorrect. Accordingly, for the five-place table in the present volume no attempt has been made to include proportional parts, but a rule has been given that will enable the interpolation to be performed mentally. This may seem somewhat troublesome, at first, to one who is not used to logarithmic computation, but after a little practice it will be found to present no difficulties. As a five-place table is needed only occasionally the arrangement of columns as in the four-place table has been sacrificed for one which allows the tabular differences to be given at more regular and convenient intervals.

I have replaced the perpetually misleading common name for the representative value of a set of residuals by one which is free from this objection and at the same time suggests the nature of the quantity in question. A few other innovations will be found scattered through the text, but for the most part the book follows well-beaten lines. The final lesson will probably seem harder than the preceding ones, to most students; but, if desirable, it can be considered as a sort of appendix, and used only for reference.

I have found it advisable to devote the first ten or fifteen minutes of the laboratory period to a rapid recitation based on the lesson of the previous day; and have allowed the students to compare many of their important numerical determinations by having them record certain specified results each day upon a large card ($22\frac{1}{2}'' \times 26\frac{1}{2}''$) that is kept on one of the

laboratory tables, and is ruled into separate columns headed by each student's name and having separate lines for each *datum*. For the fifteen exercises of this book the following *data* may be suggested:

Weights and measures: density of a (brass-and-air) weight.

Angles: relatively largest error of measured sines.

Accuracy: experimental value of π, and its error.

Logarithms: calculated value of $1 + 1 + 1/1.2 + 1/1.2.3 + 1/1.2.3.4 + \ldots$.

Small magnitudes: results and mean of a double weighing.

Slide rule: approximate ratio for π, different from 22/7.

Graphic method: least x for which $exp\ (-\ x^2)$ is indistinguishable from zero.

Graphic analysis: equation of black-thread experiment.

Method of coincidence: measured length of an inch; or slide-rule ratio of 1 gm to 1 grain.

Measurements: mode and extremes of measured variates.

Statistics: average, median, and quartiles of variates.

Dispersion: comparison of semi-interquartile range and dispersion.

Weights: weighted average for the density of aluminum.

Criteria of rejection: closer values of the ratios 10 : 12 : 15.

Least squares and errors: displacement of the second hand.

Most of the apparatus required will be found to be included in that which is used in other physical experi-

ments; a complete list of what is needed for each group of two students is given here.

2 metre sticks (graduated in tenths of an inch on the back).

2 30-cm rulers.

1 50-cm^3 graduated cylinder.

1 10-cm^3 graduated pipette.

1 platform balance or trip scale with slide giving tenths of a gram.

1 set of brass weights, 1 gm to 500 gm.

1 set of iron weights, 1 oz to 8 oz.

1 pair of fine-pointed dividers.

1 combined protractor and diagonal scale.

1 brass disc for the measurement of π.

1 ten-inch slide rule without celluloid facings but provided with A, B, C, D, S, L, T, scales, metric equivalents, and a runner.

1 hard wood block.

1 vernier caliper.

100 seeds or other variates.

1 aluminum block for density measurements.

1 set of "overflow can" and "catch-bucket" for Archimedes' Principle.

2 square wooden rods for balance pans.

1 iron clamp to hold balance on cross-bar over table.

String.

Fine black thread.

Cardboard.

Large wire nail.

Test-tube.

The student should have a watch with a second hand, a pocket-knife, and the supplies mentioned in the

introduction; a clock that beats audible seconds should be available. The slide rule should have 6745 on the C scale marked by making a shallow cut with a sharp knife and rubbing in a little oil pigment. The notebook used at the Jefferson Laboratory of Physics measures about eight by ten and a half inches and is ruled both horizontally and vertically at intervals of one-seventh of an inch.

CONTENTS

INTRODUCTION

GENERAL directions and advice in regard to study, observation, experimentation, care of apparatus, character and arrangement of notes, etc., will be given at the time of the first exercise in the course of Laboratory Physics. The student should be prepared with the following equipment:

Material Equipment.—The *note-book* should be of the size and character best adapted to the work. It can be obtained from the Department of Physics or else directions will be given as to what kind of a note-book should be used. The student will need to supply himself with a *fountain pen;* one of good quality will last long enough to make its cost less than two cents per month. A piece of *blotting paper* should be obtained which is long enough to reach across the page of the note-book. A *hard pencil* with the point kept well sharpened will also be needed.

Mental Equipment.—The student should have a knowledge of algebra as far as the solution of equations of the first degree; also a sufficient knowledge of plane geometry to include the properties of perpendiculars, equal triangles, isosceles and similar triangles, the area of parallelograms and triangles, the theorem of Pythagoras, and the properties of similar figures. An intelligent comprehension of principles is as important as a memory of rules and formulæ. He should not have

any difficulty in applying and understanding the use of letters for known and unknown quantities, symbols of operation, and parentheses; and should realize that algebraical identities are true for whatever numerical values may be substituted. He should be able to solve an equation for any of its literal components, whether they are x, y, or z, or any other letters of the alphabet. He should have a knowledge of factoring, the reduction of fractions, and the simplification of equations. The fundamental laws of exponents should be known and there should be no difficulty in dealing with negative and fractional powers. It is important in all physical calculations to be able to find mentally an approximate value for a numerical formula, and there should never be any difficulty in pointing off the product of two numbers expressed with decimals, without using the rule for the number of decimal places. It is particularly important that the student should appreciate the facts that a ratio and a quotient and a fraction are all the same thing, that a proportion is an equation in which one fraction is equal to another, that a ratio indicates that one quantity is a certain number of times as large as another, that this "number of times" is a constant for both sides of a proportion and any proportion can be written in the form

$$\begin{cases} x_1 = c_1 y_1 \\ x_2 = c_1 y_2 \end{cases}$$

if

$$x_1 : y_1 :: x_2 : y_2,$$

that x and y must be proportional if x always equals $c_1 y$ for any values of x and y, that a change in one of two quantities which are proportional must be accom-

panied by an equal relative change in the other, that two quantities which are inversely proportional have a constant product instead of a constant quotient and an increase in one is necessarily accompanied by an equal relative decrease in the other, etc. A knowledge of the rule "product of means equals product of extremes" is a very poor substitute for an *understanding* of variation and proportion, and with a proper comprehension of the subject there is no need of even knowing what is meant by such expressions as "proportion by composition and division."

The student should read each of the following questions and answer it mentally *without hesitation.* If it is necessary to "stop and think" about any of them he should make a note of the ones which cause the trouble and ask the advice of his instructor in regard to them. This will often make a great difference in the ease of performing the later laboratory work.

What is the value of $(m - x)(m + x)$?
What is the reciprocal of 2/7?
Reduce 0.395 to a percentage.
What per cent. is .005?
Write the cube of $a + x$.
Solve $3 : 12 :: 16 : x$.
Simplify $\dfrac{a/2b}{2/ab}$.
What is the fourth term of 1000 : 100 :: 31 : ?
Simplify each of the following:
$a^{7 \times 5}$, a^{7+5}, $a^7 \times a^5$, $a^7 + a^5$, $(a^7)^5$, $(a^5)^7$.
Write the value of π.
$3 + (4 - 4 \div 2)(7 \times 4 - 3) = ?$
If pv is a constant how will v be affected by doubling p?
How would your solve $3 + \dfrac{4}{x-2} = 5 + \dfrac{4x - 7}{x - 3}$?

What can you say about the value of x in the equation $3 + \frac{4}{x-2} = 5 + \frac{4}{x-2}$?

What is $x^2 - 7x + 12$ when $x = 5$?

What is the value of y in the equation, $y = ax + b$ when $x = 0$?

Write decimally, and add, the three following numbers of tenths of an inch: 9, 4, 17.

Solve $3 : 2 :: x : 100$.

Evaluate $(-2)(-15)/(-5)$.

How would you solve $1.31\,x - .44\,y = 15.2$?

State the value of $1/(1 + x)$ as a series.

Solve $p = 2\pi\sqrt{l/g}$ for l, and then for g.

Which is the larger, 37/147 or 38/148?

In the equation $\frac{x}{a} + \frac{y}{b} = 1$ what is the value of either unknown quantity, x or y, when the other is zero?

Substitute $n = -1$ in the equation $(1 + x)^n = 1 + nx + \ldots$

Substitute $n = {}^1/_2$ in the same equation and write without fractional exponents.

Solve $4^2 : x^2 :: 25^2 : 75^2$.

What is the value of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots$?

Solve $2\,x = 57°\,17'\,44''$.

What is the value of x^3/x^5. Write it with a radical sign.

Reduce $\left(\frac{1}{a^{-x}}\right)^3$ to the form a^n.

Multiply .0011 by .00011.

Evaluate $ab/c + d\,(e + f)$ when $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, and $f = 6$.

Substitute $\frac{1}{m}$ for $-\log y$ in $1/(-\log y)$ and simplify.

State the approximate square root of each of the following: 2560, 256, 25.6, 2.56, 0.256.

Simplify x^{3-2}, $x^{3\div 2}$, x^{0-3}, $4^{1/2}/4^{-2}$.

Write the following in the form of decimal fractions: $\frac{1}{3}$, $\frac{4}{5}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{2}{9}$, $\frac{1}{11}$.

What do you know about the right-hand side of the equation 5 days : 7 inches :: ?

Does $\frac{13.27 \times 0.81}{3826/41} \times 747$ have a value of about 6, or about 60, or about 600?

If $n = \frac{1}{2l}\sqrt{\frac{t}{sr}}$ how does n change if l becomes $\frac{1}{4}$ of its former size? If t becomes $\frac{1}{4}$ as large? If s is $\frac{1}{4}$ as large?

Fill out the following table:

n	n^2	2^n-3	$1/n$
0			
1			
2			
3			
$2x$			

Physical Arithmetic.—When a calculation is made with numbers obtained from physical measurements the final result never needs to be expressed with a greater number of figures than the original data contained. This means that time and labor can be saved by abridging the customary methods of "long" multiplication and division as indicated in the following examples:

```
236453)6764309(28.60741
       472906
       -------
       2035249
       1891624
       -------
        1436250
        1418718
        -------
          1753200
          1655171
          -------
            980290
            945812
            ------
             344780
             236453
             ------
             108327
```

```
236453)6764309(28.60741
       472906
       -------
       2035249
       1891624
       -------
        143625
        141872
        ------
          1753
          1655
          ----
            98
            95
            --
             3
             2
             -
             1
```

2

The abridged method is similar to the ordinary method except that the divisor and dividend are made to fit each other not by stretching out the dividend with added zeros but by trimming off the divisor. The beginner should work out the quotient of the two numbers given above, cancelling the right-hand figure of the divisor whenever he would otherwise "bring down" a zero, but not referring to the example until he has finished. After the second figure of the quotient has been obtained the problem becomes, not

$$1436250 \div 236453, \text{ but } 143625 \div 23645\cancel{3}.$$

It should be noticed that before multiplying 23645 by 6 a quick mental determination is made of the amount which should have been "carried" from the figure that was last cancelled. $6 \times 3 = 18$, giving 1 to carry, but 1.8 is nearer to 2 than 1 so it is more accurate to carry 2 instead. After the figure 4 of the quotient is obtained $23\cancel{6}\cancel{4}\cancel{5}\cancel{3}$ is to be multiplied by 4. Ordinarily it is sufficient to take the nearest cancelled figure and say $4 \times 6 = 24$, giving 2 to carry; but as 24 comes close to the number 25, which is on the boundary between 2 to carry and 3 to carry, it is well to investigate one more cancelled figure, saying $4 \times 4 = 16$; about 2 to carry; $4 \times 6 + 2 = 26$, giving 3 to carry; $4 \times 23 + 3 = 95$.

(a)	(b)	(c)
32.4761	32.4761	$3\cancel{2}.\cancel{4}\cancel{7}\cancel{6}\cancel{1}$
719.283	719.283	719.283
974283	2273327	2273327
2598088	324761	32476
649522	2922849	29228
2922849	649522	650
324761	2598088	260
2273327	974283	10
23359.5066363	23359.5066363	23359.51

Three methods of performing a multiplication are given above. Examine method *b* closely, and decide for yourself how it is carried out and whether it is perfectly justifiable; *i. e.*, whether it must necessarily give the same result in all cases as method *a*. Repeat the above multiplication as in method *b* except that after each horizontal line of partial products is obtained the right-hand figure of the multiplicand is to be cancelled and the next partial product started directly under the previous ones. Compare the result with method *c*.

Make up and practice other examples of abridged multiplication and division, some having more figures in one number than in the other. It is very important that the method shall be clearly understood, as the unabridged methods will not be permissible in any of the work done in the laboratory course.

I. WEIGHTS AND MEASURES

Apparatus.—Scale of centimetres and millimetres; graduated cylinder; graduated pipette; irregular solid; platform balance; set of weights from 1 gm. to 500 gm.; set of avoirdupois weights, 1 ounce to 8 ounces; small test-tube.

C. G. S. System.—In all scientific work the older weights and measures, such as the cubit, palm, pace, or foot, have been superseded by a single system of weights and measures which is now in universal use in all civilized countries. It is usually spoken of as the C. G. S. system, from the initial letters of the units of length (the centimetre), of mass (the gram), and of time (the second). These three units are called "fundamental," and all of the other units of the system have been so chosen as to depend upon these three in as simple a manner as possible. For example, the "derived" unit of velocity is such as will denote movement through a single unit of distance in a single unit of time; the density which is taken as numerically equal to one is the density of such a substance as will weigh one gram for each cubic centimetre of its volume; the unit of force is the force that must act for a unit of time in order to produce a change of one unit in the velocity of a unit mass.

Unit of Length.—The scientific unit of length is the *centimetre*. It is equal to about half a finger-breadth and is often to be found on tape-measures, rulers, etc.

These are simply copies of accurate standards belonging to the manufacturers, which in turn owe their accuracy to a careful comparison with the standards of the government. In the case of the governments that subscribed to the Metric Convention, including the United States, the standards, called *national prototype metres*, are lengths of one metre (100 centimetres) carefully laid off between lines near the ends of certain bars of platinum-iridium alloy which are rather more than a metre long and have a cross-section that somewhat resembles a letter X. The greater length is used instead of a single centimetre because it can be measured more accurately. The standards were constructed at Paris and distributed by lot among the signatories to the Metric Convention about 1889, after being carefully compared with one another so that their relative errors were accurately known. One of these, which is kept at the International Bureau of Weights and Measures, near Paris, is known as the *international prototype metre* and gives the same length as the original flat platinum bar constructed for the French Government by Borda and called the *mètre des archives.* The Borda standard was intended to equal one ten-millionth of the length of the meridian quadrant passing through Paris from the north pole to the equator, the earth itself thus furnishing the original standard. The metal bar, however, is now taken as the fundamental standard, not only because a microscopic measurement of it can be performed more easily and more accurately than a geodetic survey, but also because the actual length of the earth's quadrant is continually changing. Its average length,

according to the best estimations, is about 10,002,100 metres.

The multiples and subdivisions of the metre that are in actual use are the kilometre (1 km. is 1000 metres, or about $\frac{5}{8}$ of a mile), the centimetre (1 cm), the millimetre (0.1 cm), and the micron (0.0001 cm).

Units of Area and Volume.—The scientific unit of area is the *square centimetre* (1 cm^2), the area of a square each of whose sides is 1 cm. in length. A square foot is about 1000 cm^2. The unit of volume is the *cubic centimetre* (1 cm^3), the volume of a cube that measures one centimetre on each edge. The dry and liquid quarts are both in the neighborhood of 1000 cm^3.

Units of Mass and Density.—The scientific unit of mass, or for practical purposes the unit of weight *in vacuo* at sea level, lat. 45°, is the *gram* (1 gm.). It is derived from kilogram prototype standards (1000 gm. = 1 kgm. = 2.2 lbs.) established at the same time as the standards of length and was originally intended to be the mass of one cubic centimetre of water under standard conditions. More careful measurements, however, on water free from dissolved air have shown that even at the temperature of its greatest density (3.98° C.) a gram of water occupies a trifle more space than one cubic centimetre, although the excess is only one-seventieth as great as it is at room temperature. In cases where the expansion of water can be neglected it may be considered that its density, or ratio of mass to volume, is unity.

Unit of Time.—The scientific unit of time is the *second*, the 1/86400 part of the length of an average

day from noon to noon. As the length of the solar day varies at different seasons of the year the second is practically determined as 1/86164.1 of the time of a complete rotation of the earth with respect to the fixed stars. Fairly accurate seconds can be laid off by counting, at an ordinary conversational speed, "one thousand and one, one thousand and two, one thousand and three," etc.

Practice in Using the C. G. S. System.—Across the top of one page of the note-book draw a horizontal line just ten centimetres long and rule two short perpendicular lines across its ends to establish the length more definitely. Along the right-hand edge of the page, draw a line twenty-five centimetres long. Under the first line draw five others of various lengths without measuring them. After they have been drawn measure each one with a scale of centimetres and millimetres, and record its length to the nearest millimetre. For example, if the length appears to be about $172\frac{1}{4}$ mm. it should be recorded as 17.2 cm.; if about $172\frac{3}{4}$ mm. it should be called 17.3 cm.

Rule for "Rounding Off" One-Half.—If it is impossible to decide between 17.2 and 17.3 the preferred rule is to record the nearest even number rather than the odd number that is equally near. In a series of several measurements this procedure will be as apt to make a record too large as to make one too small, and the average of several such values will have only a very slight error, if any. If the rule were that the half should be regularly increased to the next larger unit the errors would not balance one another and the average would tend to be brought up to a larger value

than it should have. The same advantage could of course be obtained by always using the odd number, but the even number has one slight additional merit, namely, that in case it should have to be divided by two a recurrence of the same situation would be avoided.

By comparison with the lines already drawn make a mental estimate of the length and width of the notebook; then verify the estimate by measuring with a scale. Remember to record clearly all experimental work that is done; thus, the completed notes should show at a glance which number is the actual width and which is the rough estimate.

The Hand as a Measure.—Hold your hand across a centimetre scale and either spread the fingers slightly or crowd them closer together, as may be necessary, so as to make a whole number of finger-breadths equal to a whole number of centimetres. Then hold the hand in a similar position while using it for practicing approximate measurements of various objects. Record also the exact measurements of the same objects as they are obtained afterward with the graduated scale.

Separate the thumb and little finger as far as can be conveniently done without special effort and measure your span in centimetres and .millimetres. Repeat this measurement five times, being careful not to let the sight of the scale under your hand influence the extent to which the fingers are spread, and decide which is the most satisfactory value. Measure the breadth of the table by means of successive spans, using finger-breadths for the final fraction of a span, and compare the result with the actual breadth.

Measurement of Area.—Ask the instructor to draw an irregular outline in your note-book. Count the number of squares of the ruled paper which are entirely included within it, and to this add half the number of the squares that are cut by the boundary of the figure. The result will be the area of the irregular figure, not in square centimetres, of course, but in terms of the small ruled squares. Try to obtain a more accurate value for the total area by estimating as closely as possible how many tenths of each small square which is cut through by the curve are included within it. Why does the first result agree so closely with the second?

On the same irregular figure block off an equal area by drawing several rectangles and triangles to cover it. If a corner of a triangle projects considerably beyond the irregular line draw one side of the triangle so that it includes less than the requisite amount and try to make the two opposite errors balance as nearly as possible. Then find the total area of the geometrical figures, measuring their dimensions not by means of the centimetre scale but with a scale copied from the ruling of the note-book or by transferring each length to the ruled page with a pair of dividers, so as to obtain the result in the same units as before.

Measurement of Volume.—Examine the graduated cylinder and pipette; compare the indicated volumes with an imaginary cube built up on one centimetre of the graduated scale. Pour into a test-tube an amount of water which you consider to be 10 cm^3; then measure it carefully.

Put an irregular block of metal into a graduated

cylinder partly filled with water and determine its volume by the change in the water level. In reading a cylinder or a pipette the most accurate result is obtained by noting the height of the lowest part of the surface of the liquid.

Measurement of Mass.—Examine the brass weights in a set extending from 1 gram to 500 grams, and observe especially the size of the 10-gram weight. What would its volume be if its density were ten? Is its volume greater or less than this if the density of brass is 8.5?

Examine the platform balance and notice the two wooden wedges that raise the pans so as to keep their weight off from the accurately ground bearings when the apparatus is not in use. Remove the wedges carefully and notice where the moving pointer finally comes to rest on the scale. Notice the counterpoise, which can be screwed toward one side or the other in order to adjust the position of equilibrium, but do not attempt to move it unless you are sure that the apparatus is level and the scale pans are clean. Note the sliding weight that is used for weighing fractions of a gram.

Find the mass, in grams, of a four-ounce avoirdupois weight, and determine for yourself why it is advisable to put the object to be weighed on the left-hand scale pan rather than on the right-hand one.

Measurement of Density.—Find the volume of the 200-gram weight by measuring its height and diameter as carefully as possible, but do not immerse it in water. Imagine the handle of the weight to be flattened out and spread over the top of the cylindrical part and

estimate as well as you can how much this would add to the height of the cylinder. Then calculate the density of the *weight,* using the abridged methods of multiplication and division. The result may turn out to be less than the usual density of brass if the weight consists of two parts with an air-space between them.

Find the mass also of the irregular solid whose volume was determined by immersion, and determine its density.

II. ANGLES AND CIRCULAR FUNCTIONS

Apparatus.—A pair of dividers; a protractor provided with a "diagonal scale."

Unit of Angle.—The practical unit of angle is the *degree,* and a right angle contains 90 of these units. Each degree is subdivided into sixty minutes, and each of these is in turn made up of 60 still smaller units, or seconds. In scientific usage, however, an angle of a given size is considered, not as a number of units, but as the ratio of one such number of units, which represents a certain length, to another number of units, which represents a different length. This means that it is not expressed in any arbitrary concrete unit whatever, but as an abstract number. The reason for this can be understood if a length of 1 centimetre and an angle of 90 degrees are drawn on a sheet of paper and observed through a magnifying glass. The centimetre may now appear to be two centimetres in length, but the angle of 90 degrees does not become 180 degrees, it remains exactly as large as before. The same thing is of course true of a pure number: with

a very slight magnification two objects may both be made to look larger, but no amount of magnifying power will make them look like three.

Numerical Measure of an Angle.—If a circle is drawn with the vertex of any angle as its centre the arc intercepted by the angle will have a certain length as compared with the radius, and this ratio of arc to radius will be the same whether the circle is large or small. It will depend only upon the size of the angle, and so can be used as a measure of it. The angle, then, which is equal to the number one must be such an angle that the corresponding arc and radius are of the same length. It is also clear that an angle which extends entirely around a point, that is, four right angles or 360 degrees, must have 2π for the ratio of its arc, the circumference, to the radius; and if 2π is equal to $360°$ the angle π must be $180°$, and the angle 1 will be $180°/\pi$.

Practice translating such numbers as the following into degrees until it can be done without any hesitation:

$$\tfrac{1}{2}\pi = ? \quad 2\pi = ? \quad \tfrac{1}{4}\pi = ? \quad 4\pi = ? \quad \pi/3 = ? \quad \tfrac{3}{2}\pi = ?$$
$$\pi/4 = ? \quad \pi/\pi = ?$$

Work out the number of degrees in the unit angle by the abridged method of division. Is the *angle 3* greater or less than 180 degrees?

Construction and Measurement with the Protractor.—Examine the protractor and the scale of degrees that is engraved on it. Notice the line joining $0°$ and $180°$, and decide exactly where the vertex of the indicated angles must be located on the instrument.

Draw a horizontal line in the note-book and mark

any point on it with a short cross-line. How can you use the protractor to draw a line which passes through this point and makes a given angle with the horizontal line?

Draw a triangle whose angles are $\pi/2$, $\pi/3$, $\pi/6$. Draw another so that its angles are $\pi/4$, $\pi/4$, $\pi/2$.

The Diagonal Scale.—If the protractor is provided with what is known as a diagonal scale notice that at the bottom of this scale there is a horizontal line which is divided into centimetres or inches, and that one division at the end of the scale is subdivided into millimetres or tenths of an inch. Notice that any number of centimetres and tenths, within limits that depend upon the total length of the scale, can be found already laid off as a single, continuous stretch of the base line, the millimetres being measured from the proper point to the junction of the millimetre scale and the centimetre scale, and the centimetres then extending onward the required distance beyond the junction point.

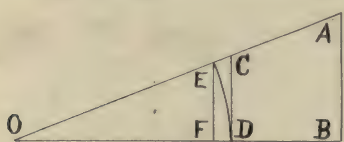
Apply a pair of fine dividers to the ends of some object that is not longer than the diagonal scale; then transfer them to the scale and measure their spread to the nearest millimetre. They should always be held flat to avoid marring the scale.

Notice that there are ten other lines, spaced at equal distances and parallel to the base line, and that they are intersected by certain diagonal lines. Observe one of these that forms the hypothenuse of a large right-angled triangle and determine the length of its shortest side. If the parallel lines are equidistant what are the lengths of their short segments that are

included in the large triangle and cut it up into smaller similar triangles? Determine for yourself how it is that this arrangement will show any number of centimetres and tenths and hundredths of a centimetre as a single distance on one of the horizontal lines, laid off between two intersecting lines. Practice using the dividers and diagonal scale, both for laying off predetermined lengths and for measuring unknown distances, until you are perfectly familiar with the method of procedure.

An angle may be identified or measured in other ways than by the number of degrees that it contains or by the ratio of arc to radius. In case one side of the angle is horizontal a given slope or inclination of the other side will always correspond to a definite angle. The usual method of indicating a slope is by means of the ratio of height to base-line, or amount of vertical distance for a unit amount of horizontal distance.

The Tangent of an Angle.—Thus, in the figure, the slope OA is numerically given by the ratio $AB:OB$, or $CD \div OD$, or EF/OF —all obviously equal to one another and to about 0.39—and it is said that OA has a 39 per cent. slope, or that



the tangent of $\angle\ AOB$ is 0.39. The reason for calling this quantity the "tangent" of the angle may be understood by noticing that the line drawn between the two sides of the angle and *tangent* to the arc at one end will be numerically equal to the ratio in question if the radius, OD or OE, is of unit length, for then $CD/OD = CD/1 =$

CD. If the radius is not of unit length the tangent line will of course be greater or less than the numerical tangent of the angle. In general, if A is an acute angle of a right-angled triangle the ratio of the side opposite this angle to the adjacent side is called the tangent of the angle A, and is customarily abbreviated to *tan A*.

With the protractor draw carefully an angle of 58 degrees, and a perpendicular to one of its sides. Measure the tangent and see how close your experimental result comes to the theoretical value 1.600. What is the numerical value of *tan* 45°? *tan* 0°? *tan* $\pi/4$? Draw angles of 80°, 85°, 88°, 89°, remembering that the base line can always be made short enough to bring the perpendicular within the limits of the paper; then decide what the numerical value must be for *tan* 90°. How could you find the tangent of the angle *OAB* on page 30? What is its approximate value? What relation is there between the numerical values of *tan OAB* and *tan AOB*?

Table of Tangents.—Near the bottom of the next unused page of your note-book draw a horizontal line of 20, 25, or 50 squares in length, following one of the blue ruled lines and extending nearly across the page. Call its length unity (1), erect a perpendicular at its right-hand end, and on the perpendicular lay off heights equal to *tan* 10°, *tan* 20°, *tan* 30°,..., as far as the length of the page will permit, using the values obtained from the table on page 141. Opposite the number of degrees in the column headed DEG will be found the corresponding tangent, with the decimal point omitted, in the column TAN. From 45° to 90°

the numbers run upward in the right-hand columns over DEG and TAN at the bottom of the page. If there is any trouble in understanding the arrangement of the table use it to verify the following equations before proceeding further:

Tan $1° = .0175$; *tan* $2° = .0349$; *tan* $6° = .1051$; *tan* $44° = .9657$; *tan* $45° = 1.000$; *tan* $46° = 1.036$; *tan* $84° = 9.514$; *tan* $85° = 11.43$; *tan* $89° = 57.29$.

Draw slant lines from the points marked off on the vertical line to the left-hand end of the base line and test the angles with the protractor to see that they are 10°, 20°, ... If mistakes have been made repeat the construction on the next page; do not correct the first diagram by erasures.

The Sine of an Angle.—With the base line as a radius draw an arc about its left-hand end extending from 0° to 90°. Complete the series of angles to 90° by laying off ten-degree arcs with a pair of dividers. Find the point where the arc intersects the line whose slope is 10 degrees, and measure carefully the vertical distance from this point down to the base line. If the diagram has been carefully drawn the value should come out 0.174; remembering always that the unit of measurement is not the small square but the whole base line. This number is less than *tan* 10°, and corresponds to EF/OD in the previous diagram, or to its equivalent, EF/OE. This ratio of height to slant distance is another quantity which is perfectly definite for a given angle and is called the "sine" of the angle. For any angle not greater than 90° we may define "*sin A*," as it is usually abbreviated, as the ratio of

the opposite side to the hypothenuse, in a right-angled triangle similar to the one used for defining *tan A*.

In the same way measure *sin* 20°, *sin* 30°, ... *sin* 90°, or as many of them as may be directed by the instructor; and tabulate the results as shown in the margin, where *sin* 20° is represented as having been measured and found to be .344, and then corrected by comparison with the table on page 141, which gives it as .344 − .002, or .342. Notice that the sine, like the tangent, is to be read upward from the bottom of the page for angles greater than 45°. After your table of sines has been completed find the angle whose sine has the largest correction, and divide this correction by the true value of the sine in order to find the relative error of your measurement. Thus, if the quotient is .0024 your measurement has an error of 24 parts out of 10,000, or about $\frac{1}{4}$ of 1%.

angle	*sine*
0	0
10°	.174 ± 0
20°	.344 − .002
30°	.500 ± 0
40°	.640 + .003

Definition of Function.—The sine and the tangent of an angle are called "functions" of the angle, the expression *function of a variable magnitude* meaning simply a second quantity which has a definite value for each value of the first magnitude. For example, $x^2 - 3x + 2$ is called a function of x, because it has a definite value for any definite value which may be assigned to x. Similarly $\sqrt{x}$, logarithm of x, a^x, *tan* x, are all functions of x, as is also any other algebraical formula which involves x.

The Cosine of an Angle.—The only other function of an angle, or "circular function," in frequent use is

3

the *cosine*, corresponding to OF, or more strictly to OF/OE, in the figure above. It is the ratio of adjacent side to hypothenuse, and in the note-book diagram is the left-hand segment of the base line which is cut off by the vertical sine-line.

From the diagram in your note-book find the angle whose cosine is $\frac{1}{2}$ by locating the point on the arc that is directly above the centre of the base line. What is *cos* 0°? *cos* 90°? Verify your statements by reference to the table on page 141.

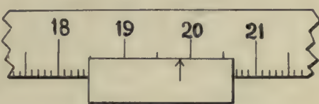
III. ACCURACY AND SIGNIFICANT FIGURES

Apparatus.—Scale of centimetres and millimetres; card or strip of paper; circular brass disc for measuring a circumference.

Estimation of Tenths.—In the physical laboratory all measurements are to be made as accurately as possible unless there is some evident reason for making them more roughly. Measurements with the metre stick should, of course, be expressed to the nearest millimetre rather than the nearest centimetre. This is not the limit of accuracy, however. With a little care it is not difficult to imagine each millimetre divided into ten equal parts and to estimate just how many of these parts are included in the length that is to be measured. Experienced observers even attempt to determine hundredths of a millimetre in the same way, and find that one man's estimate will hardly ever differ from another's by more than one or two hundredths. The estimation of hundredths of the smallest scale divisions, however, is chiefly of import-

ance to the expert; the beginner will find that even the estimation of tenths is a rather uncertain process, but proficiency can be gained rapidly by experimenting with the larger subdivisions of a scale, where the determinations can afterward be verified.

Draw a short line at right angles to the edge of a card or slip of paper and hold this edge on a scale of centimetres and millimetres in such a way that the smallest graduations are hidden but the marks indicating centimetres and half-centimetres are visible. Mentally divide the half-centimetre in which the cross-line



comes into five equal parts by four imaginary lines, and in this way estimate the scale reading to the nearest millimetre. In the figure the line seems to be either three fifths or four fifths of the way from the scale-division 19.5 to 20.0, but it may be difficult to decide which is the nearer without actual measurement. In practice, however, the estimate should be written down and the card then be allowed to slide carefully across the ruler until the millimetre scale is just exposed. The estimate is afterward to be verified or corrected, as the case may require.

Make ten such estimates with the card placed anywhere along the scale at random, tabulating the determinations and the verifications in parallel columns. Then hold the card a trifle higher, so as to hide the half-centimetres as well as the millimetres and make twenty more determinations; these will also be estimates of the nearest millimetre, but will require the

more difficult process of deciding upon tenths of a whole centimetre instead of fifths of a half-centimetre.

Mistakes in Estimating Tenths.—Omitting the preliminary estimate of fifths, examine the table of estimated tenths closely and find out what kind of error you are most apt to make. Some students find it hardest to estimate 0.3 and 0.7 correctly; others have almost a uniform tendency to read a position like 12.0 as either 11.9 or 12.1. The latter mistake is due to the fact that a minute deviation from the position of a visible graduation is very easily noticed and there is a tendency to consider it as a single tenth. Of course if it amounts to more than half of a tenth this is correct; but if it is less than half a tenth it should be considered as 0.0 instead of 0.1. The same bias may even cause a tendency to read 0.1 and 0.9 as 0.2 and 0.8. On the other hand there may be just the opposite error if the graduated lines are rough or coarse, unless the student is careful to estimate from the imaginary centre of such a "line" instead of from its margin.

If a definite kind of error is evident from a study of your table see if it can be overcome when making another short series of determinations. Then draw a second line on the card, place the latter in position as before, estimate both points, and find the distance between them by subtraction. This is the customary method of measuring a length, and is preferable to making one line coincide exactly with a scale division and estimating only the other one, in spite of an obvious additional source of error.

Value of π.—Make an experimental determination of the value of the constant, π, by rolling the brass disc along the graduated surface of a metre stick in order to determine its circumference by noticing the readings, to a tenth of millimetre, where the marked radius touches the scale at the beginning and the end of the measurement. Measure the diameter with especial care, noting down centimetres, tenths, and hundredths; divide the circumference by it, and see how many figures of your result agree with the theoretical 3.141592653589793238462643383279502884197169.

Physical Measurement.—The operation of making a measurement is merely counting; it is the determination of how many units are required in order to be equal to a given quantity of the same kind. But while a count such as a census of the number of individuals in a town must give a perfectly definite whole number it usually happens that a physical measurement will not give a whole number, or even a commensurable number except as the result of an error, and successive repetitions of a measurement will give a number of different apparent values. Accordingly, any numerical statement of a measurement must be merely an approximation to an unknown truth, and will be perceptibly correct or perceptibly incorrect according to how closely it is examined.

Ideal Accuracy.—A considerable part of the student's difficulty in grasping the matter of relative accuracy lies in the facts that he has usually had very little practice in careful measurement and that his previous study of arithmetic has emphasized a condition of infinite accuracy of numerical values. Such a number

as 12.5 is supposed not only to mean 12.50, but also to be equal to 12.500000 . . . to an unlimited number of decimal places. This is quite proper and satisfactory as long as one realizes that he is dealing with imaginary quantities, or perhaps it would be better to speak of them as ideal quantities, perfections of measurement which have no more real existence than the point, line, plane, or cube, of the geometrician. The smooth surface of a table does not come as near to being a plane as does the surface of an optically worked block of glass or a Whitworth plane, and even the smoothest possible surface can be magnified so as to show that it contains irregularities everywhere. Perhaps if it were magnified enough we could see that its shape was not even constant, but individual molecules would be found swinging back and forth or even escaping from the surface. A geometrical plane certainly corresponds to nothing in reality, outside of the imagination, and perfect accuracy of number is just as much an imaginary concept.

Decimal Accuracy.—If 12.5 cm, as a measurement, does not mean 12.50000 . . . to an infinite number of decimal places what does it mean? As different measurements are often made with different degrees of accuracy the universally adopted convention is merely the common-sense one that *the statement of a measurement should be accurate as far as it goes, and should go far enough to express the accuracy of the determination.* Thus 12.5 cm means that a length is nearer to precisely 12.5 than to precisely 12.4 or 12.6 cm; 12.50 cm, however, implies that the stated length is nearer the exact value of 12.5 than to either

12.49 or 12.51, that is, that it has been measured to the nearest tenth of a millimetre and does not differ from twelve and a half centimetres by more than .005 cm, while the expression 12.5 cm states merely that the length is between $124\frac{1}{2}$ and $125\frac{1}{2}$ millimetres. If the length had been 12.5 cm (*i. e.*, any of the following: 12.46, 12.47, 12.48, 12.49, 12.50, 12.51, 12.52, 12.53, 12.54 cm) the statement that it was 12.50 cm would violate the requirement that a measurement should be accurate as far as it goes, for the chances are eight to one that it would be one of the other numbers of hundredths given above. On the other hand, if an observer determined a length to be 12.50 cm and only stated it as 12.5 cm he would not be doing justice to his own accuracy. The principle is simple enough when it has once been realized, and then there is little danger of the student's "rounding off" a carefully obtained measurement like 2.638 gm. to 2.64 gm. merely for the sake of doing some rounding off; but he will probably find it necessary for some length of time to be careful not to cut off a final significant zero. If the brass π-disc is found to be "just 8" centimetres in diameter the measurement should be stated as 8.00 cm if tenths of a millimetre were estimated and none found; but it should be given as 8.0 cm if the student read the millimetres but was unable to estimate smaller amounts. There is nothing to show which degree of accuracy was obtained if the diameter is put down as 8 cm "because it came out just even."

Significant Figures.—The figures of which a number is composed, if they are all necessary to express its

accuracy, are called *significant figures*, but ciphers added to the right of the other figures merely to fill out a column are not significant, nor are ciphers on the left of a number, such as 0.75 or 050. Neither are ciphers significant when used merely to locate the position of the decimal point, whether on the left, as in the statement that the length of a certain light-wave is .00005086 cm, nor on the right, as when the sun is stated to be 93000000 miles from the earth; the first of these numbers has four significant figures, the second only two. It will be noticed that with a number like the last there is trouble in applying the rule of making it "accurate as far as it goes." Only the first two or three figures are known, but eight are needed in order to place the decimal point. Moreover, suppose the distance should be found to be 93.000 millions of miles at some particular time; how can we make a distinction between 93,000,ooo with five significant figures, and 93,ooo,ooo with only two, unless we employ two different symbols for a cipher, one to be used when it is significant and the other when it is not? (The solution of the difficulty will be found further on in this lesson.)

The length of an inch has been determined to be between 25.3997 and 25.3998 mm. How many of the following statements are correct and how many are positively wrong?

1 in. = 25.4 mm.	1 in. = 25.40 mm.
1 in. = 25.400 mm.	1 in. = 25.4000 mm.

Which of the following values of π are correct?

3.141 3.142 3.1415 3.1416 3.141600 3.141592

Relative Accuracy.—The relative accuracy of a number depends upon two things: how much its absolute difference from the truth amounts to, and how large the number itself is. If two points on the earth's surface are found by careful surveying to be 10 miles apart the determination of distance may easily be in error by more than a foot, and even with the most extremely careful triangulation the error is likely to be as much as four inches. An error of a quarter of an inch, however, in measuring the thickness of a door could hardly be made even with the clumsiest of measuring apparatus. It is plainly misleading to say that the latter determination is more accurate than the former because $\frac{1}{4}$ in. is less than four inches. The size of the error means nothing unless we also know the size of the measurement in which it occurs. Suppose the thickness of the door is $1\frac{1}{2}$ inches, then the error amounts to one sixth of the whole measurement, or is an error of about 16%. An error of four inches out of ten miles, however, is roughly an error of one out of a hundred and fifty thousand, or about six per million, or about .0006 of 1%.

Calculation of Relative Errors.—The relative error of a measurement is not usually needed with very great accuracy. Where numbers are as different as 6 in tens-of-thousandths' place and 16 in units' place the location of the decimal point is the only thing of importance and the numbers in the corresponding situations can almost be ignored. Consequently, a rough calculation of a relative error is all that is necessary, and this can be performed mentally. Thus, in the above example, 1 foot is 1/5280 of 1 mile, hence

it is about 1/50,000 of 10 miles; and 4 inches, being $\frac{1}{3}$ of 1 ft., is about $1/(3 \times 50{,}000)$, or 1/150,000. Fifteen is about $\frac{1}{6}$ of 100, so 150,000 is about $\frac{1}{6}$ of 1,000,000; and $1/150{,}000 = 6/1{,}000{,}000$ or .000006 or .0006%.

If a length is stated as 174.2 cm. the inference is that it is nearer to that exact amount than to 174.1 or 174.3, namely, that its error certainly is not as much as 0.1 out of 174.2; this means that it is not as much as 1 out of 1742, or 1 out of nearly 2000, or 5/10000, or .0005, of .05%.

Copy the following numbers and write beside each one its approximate accuracy; *e. g.*, 66.7 is stated with an accuracy of 0.1 out of a total of 66.7, or of one part in 667, or about $1\frac{1}{2}$ per 1000, or $.001\frac{1}{2}$, or .15%. Which column has its numbers stated with about the same degree of accuracy, the one in which each number is given with the same number of decimal places, or the one in which each has the same number of significant figures?

7.23	7.23
94.07	94.1
0.51	0.512
428.00	428.
66.67	66.7

What was the relative error of your measurement of the width of the table by spans and finger-breadths in Lesson I?

In Lesson II, when the relative error of a measured sine was determined did you pick out the largest relative error by choosing the largest numerical error?

Rule for the Relative Difference of Two Measurements.—When the theoretical value of a number is known the numerical error is divided by the true value, as we have seen. When there is no theoretical standard and there are simply two experimental determinations

to be compared with each other the accepted procedure is to *divide the difference by the greater value.* Apply this rule to your two measurements of an irregular area in Lesson I, obtained by counting squares and by constructing geometrical figures. How much relative difference is there in the results of the two methods.

Calculate 719×327 and 719×325; if the third figure of one factor changes how many figures of the product can be relied upon to remain unchanged? Compare the last result with the example on page 18; if the factors are stated correctly to three figures how many figures of the product will be correct? Decide mentally, by induction, how many figures of a product can be trusted if one of its factors contains six figures and the other one has only three.

Remembering that the accuracy with which a quantity is expressed depends not upon the number of decimal places but upon the number of significant figures and keeping in mind the fact that the number of trustworthy figures in a product is the same as the number in its least accurate factor, turn back to your notes on physical arithmetic and observe that the method of abridged division automatically gives just the number of figures in the quotient that are needed if no figures of the dividend are "brought down"; and that abridged multiplication always gives at least as many as are in the smallest factor. It will not give any superfluous figures if the larger factor is used as the multiplier, but will give as many as the larger factor contains if that is used as the multiplicand.

Of course the best method is to round off the larger factor before multiplying, so that it has no more figures than the smaller one.

Standard Form.—To avoid a long string of figures when writing very large or very small numbers it is customary to divide a number into two factors, one of them a power of ten. Thus .000000017 and 632000000000 are the same as 17×10^{-9} and 632×10^{9} respectively. This notation also makes it possible to write 93000000 unequivocally with two significant figures or with four, as may be desired, for we can put it either in the form 9.3×10^{7} or in the form 9.300×10^{7}. The same value and accuracy for the second of these would be retained just as well by writing 93.00×10^{6} or 930.0×10^{5}, but it is customary to choose the power of ten so that the other factor shall have just one significant figure to the left of the decimal point. The number is then said to be written in *standard form.*

Write the following numbers in standard form:

2946.3; 0.051 (*Ans*: 5.1×10^{-2}); 666.6; .0004; 18.27; 17.042; 1.12; 25.4; 25.400; 186000.

Write a definition of standard form in your own words.

IV. LOGARITHMS

In the following equations the constant 10 is called a "base" and any exponent is called the "logarithm" of the right-hand side of the equation; thus, 3 is said to be the logarithm of 1000, or, as it is usually abbreviated:

$$3 = log\ 1000.$$

$$\begin{aligned} 10^5 &= 100000 \\ 10^4 &= 10000 \\ 10^3 &= 1000 \\ 10^2 &= 100 \\ 10^1 &= 10 \\ 10^0 &= 1 \\ 10^{-1} &= .1 \\ 10^{-2} &= .01 \\ 10^{-3} &= .001 \end{aligned}$$

Properties of Logarithms.—Those logarithms whose base is 10 are called "common" logarithms. For theoretical purposes "natural" logarithms are sometimes used; their base is "e," approximately 2.7, or $1 + 1 + 1/2 + 1/2.3 + 1/2.3.4 + 1/2.3.4.5 + \ldots$ Notice from the equations given that $log\ 100 + log\ 1000 = log\ 100{,}000$. This suggests the general relationship that

$$log\ a + log\ b = log\ (a \times b)$$

Try numerical values for the following also:

$$log\ a - log\ b = log\ (a/b)$$
$$n \times log\ a = log\ (a^n)$$
$$(log\ a)/n = log\ \sqrt[n]{a}$$

These four equations express the fundamental principles that in the use of logarithms "addition takes the place of multiplication, subtraction of division, multiplication of raising to a power, and division of root extraction."

The advantage of using 10 for a base is that

$$log\ (10 \times a) = log\ 10 + log\ a = 1 + log\ a$$

and in general

$$\log(10^n \times a) = n \log 10 + \log a = n + \log a;$$

for example, $\log 365 = 2 + \log 3.65$. Accordingly tables of common logarithms are made out only for numbers between 1 and 10, the logarithms of all other numbers being self-evident from these.

The logarithms of numbers other than powers of 10 are in general incommensurable and are given only approximately in tables. Using the small table given here find 2×3. Answer: $\log 2 = .301$; $\log 3 = .477$; their sum, .778, is found from the table to be $\log 6$, or, as it is said, the *antilogarithm* of .778 is 6. Find 2×4; 2^2; 3^2; 4×5; $\sqrt{9}$; 5×6; 50×6; 50×600. Calculate the value of e from the series on page 45. Examine the four-place logarithm table on pages 142 and 143, and notice that it contains the same succession of numbers, from 1.00 to 9.99, and of logarithms, from .0000 to .9999, as in the table on this page, but with intermediate values at smaller intervals, and without any decimal points. Verify the following statements by finding the logarithm in line with the first two figures in the left-hand column and in the column headed by the third figure:

no.	*log.*
1	.000
2	.301
3	.477
4	.602
5	.699
6	.778
7	.845
8	.903
9	.954

$\log 3.65 = .5623$; $\log 3.66 = .5635$; $\log 4.06 = .6085$; $\log 7.70 = .8865$; $\log 77.0 = 1.8865$; $\log 0.77 = -1. + .8865$; $\log .00077 = -4. + .8865$, or, as it is generally written, $\log .00077 = \bar{4}.8865$, the minus sign being written over the 4 to indicate that it applies only to the whole number, and the decimal part, or "mantissa" as it is sometimes called, being always positive.

Write the integral part, or "characteristic," of the logarithm of each of the following: 5441 (*Ans.:* 3); 27; 79264; 264; 73; 0.73; 0.073; 0.000073. Make up a rule for finding the characteristic of any logarithm.

Write the logarithms of 984; 982; 981; 980; 98; 9.8; .98; .098; 7; 14. Add the last two and find the result in the body of the table; see what marginal number is opposite it, and verify by multiplying 14 by 7.

Use of the Table of Logarithms.—A four-place table is in general satisfactory for obtaining the logarithm of a number that has as many as four significant figures; for five-figure accuracy a five-place table is necessary, etc. The process of finding the logarithms of numbers lying between two consecutive tabular numbers will be easily understood from the following example: Find $\sqrt{3.142}$. *Log* 314 = 4969; *log* 315 = 4983; difference = 14; a number that is two tenths of the way from 314 to 315 will have a logarithm that is two tenths of the way from 4969 to 4983; 0.2 of 14 = 3 (to the nearest integral value); 4969 + 3 = 4972. *Log* 3.142 = .4972; $\therefore$ *log* $\sqrt{3.142}$ = .4972 ÷ 2 = .2486. The logarithm 2486 does not occur in the table but is between 2480 (= *log* 177) and 2504 (= *log* 178); 2486 is 6/24 of the way from 2480 to 2504, and hence is the logarithm of $177\frac{6}{24}$ or 17725 (better say 1772, as the fifth figure is liable to be incorrect). Ans.: 1.772.

Notice that the small multiplication tables at the side of the main table enable the multiplications and divisions to be performed mentally.

Practice finding reciprocals *mentally* by the process illustrated in the following examples: Required, the

value of $1 \div 3.142$. *Log* $3.142 = .4972$; $0 - .4972 = .5028$, subtracting, from left to right, each figure from 9, except the last, which is subtracted from 10; $.5028 =$ *log* 3183; answer, pointed off by inspection, .3183. The value of $\log 1 - \log x$, or $0 - \log x$, is known as the *co-logarithm* of x.

Satisfy yourself that the following reasoning is correct:

If we assume that

$$y = e^{-x^2}$$

then, taking logarithms of each side,

$$\log y = -x^2 \log e$$

and

$$-\log y = x^2 \log e;$$

taking logarithms again

$$\log(-\log y) = \log(x^2) + \log(\log e)$$

or

$$\log(-\log y) = \log x^2 + \bar{1}.6378.$$

Using the last equation and logarithm tables find x^2, $\log x^2$, $\log x^2 + \bar{1}.6378$, $\log(-\log y)$, $\log y$, and y for each of the following values of x: 0, .2, .4, .6, .8, 1.0, ... 2.6, 2.8, 3.0, 4, 5. On the next unused left-hand page of your note-book tabulate the results in columns headed x, x^2, $\log x^2$, etc. Leave the opposite right-hand page vacant until after studying graphic representation.

x	x^2	$\log x^2$	$\log x^2 + \bar{1}.6378$	$\log(-\log y)$	$-\log y$	$+\log y$	y
0	0	$-\infty$	$-\infty$	$-\infty$	0	0	1
.2	.04	$\bar{2}.6021$	$\bar{2}.2399$	$\bar{2}.2399$	.0174	$\bar{1}.9826$	.961
.4	.16	$\bar{1}.2041$	$\bar{2}.8419$	$\bar{2}.8419$	.0695	$\bar{1}.9305$	.852
.6							
.							
.							
.							
3.0							
4.0							
5.0							

V. SMALL MAGNITUDES

Apparatus.—Platform balance; set of weights from 1 gm to 500 gm, set of avoirdupois weights from 1 oz. to 8 oz.

Negligible Magnitudes.—The way in which small magnitudes enter into physical calculations may be seen in the following example:

Suppose that a metal cube has been constructed accurately enough to measure 1.00000 cm along each edge. If it was brought from a cold room into a warm room a delicate measuring instrument might show that the change of temperature had increased each dimension to 1.00012 cm, and by unabridged multiplication it would be easy to prove that the area of each side was 1.0002400144 cm^2 and that the volume had become 1.000360043201728 cm^3. If the most careful measurements just allow us to distinguish units in the fifth decimal place then tenths of those units (represented by the sixth decimal place) would be impossible to measure, and the attempt to state not only tenths, but hundredths and thousandths of those units becomes absurd. By noticing that the number 1.0002400144 differs from the value 1.00024 that would be obtained by abridged multiplication only as much as one or two thousandths of the smallest measurable amount we can see clearly why the area of a 1.00012-cm square is and must be 1.00024 cm^2. Similarly, the volume of the cube is neither more nor less than 1.00036 cm^3, and the string of figures running out ten decimal places further is absolutely meaningless.

It will be noticed that 1.0002400144 is in the same form as $1 + 2x + x^2$, the square of $(1 + x)$, where $x = .00012$; also, that 1.000360043201728 corresponds to $1 + 3x + 3x^2 + x^3$, the cube of $(1 + x)$. In other words, when dealing with the objects of the real world which is evident to our senses it may happen that a measured amount is so small that its higher powers, algebraically speaking, are minute beyond all perception. Of course this must not be understood as meaning that the cube of a measurable length can ever be an impalpable volume; the cube of $1 + x$ is even a larger number, $1 + 3x$; it is the *difference* between this "physical" value, $(1 + x)^3 = 1 + 3x$, and the mathematical value, $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$ that eludes perception on account of x^2 being extremely small *in comparison with* x, which is itself minute.

The examples that have been given above suggest that if x is small enough $(1 + x)^2 = 1 + 2x$, $(1 + x)^3 = 1 + 3x$, and in general $(1 + x)^n = 1 + nx$. The matter can be tested by making use of the binomial theorem, that

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{1 \cdot 2} x^2 \pm \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \dots$$

This shows that $(1 \pm x)^n = 1 \pm nx$ if x is so small that x^2, x^3, etc. are negligible, the only possible exception being in case n should be so large that it could counterbalance the small size of x and prevent the term

$$\frac{n(n-1)}{1 \cdot 2} x^2$$

from becoming negligible. In physical measurements, however, n is never large enough for this to happen.

Properties of Deltas.—The small quantities which we have been considering are usually denoted by the Greek letter δ (read "delta"), and one of the most important properties of deltas is

$$(1 + \delta)^n = 1 + n\delta.$$

It is advisable to learn such an equation in the form given here rather than in the equivalent form $(1 \pm \delta)^n = 1 \pm n\delta$, on account of possible confusion in applying equations containing more than one double sign. Of course δ can be considered as having a negative value when necessary.

Find the square of 0.97. *Ans:* $0.97 = 1. - .03; = 1 + (-.03)$; $0.97^2 = 1 + 2(-.03) = 1. - .06 = .94$, correct to as many significant figures as are given.

Find $(1.00012)^4$ mentally.

Find $(.99988)^4$ mentally.

By algebraical division:

$$1/(1 + x) = 1 - x + x^2 - x^3 + x^4 - \ldots,$$

whence

$$\frac{1}{1 + \delta} = 1 - \delta$$

Divide 1 by .995. *Ans:* $1/(1. - .005) = 1. - (-.005) = 1.005$.

Find mentally the reciprocal of 1.00012.

Find $1/(1.00012)^2$ by using first one formula and then the other.

Find $(1.00012)^{1/2}$, and complete the following formula:

$$\sqrt{1 + \delta} = 1 + \tfrac{1}{2}\delta$$

By ordinary multiplication $(1 \pm x)(1 \pm y) = 1 \pm x \pm y + xy$, and if both x and y are small their product xy will be negligible, so that

$$(1 + \delta_1)(1 + \delta_2) = 1 + \delta_1 + \delta_2$$

Find $1.00012 \times .99890$. *Ans:* $(1 + .00012)(1 + (-.00110)) = 1 + .00012 - .00110 = 1. - .00098 = .99902$.

Find (1.0021) (1.0037) in three different ways, writing out the complete work in your note-book: (*a*) by ordinary multiplication, (*b*) by abridged multiplication, (*c*) by the use of deltas.

A barometer reading is corrected for temperature errors by multiplying it by 1.00037 and dividing by 1.00364; what is the percentage difference between the corrected reading and the original reading? Suggestion: call the original height unity.

Write the formula for $(1 + \delta_1)/(1 + \delta_2)$.

If two numbers are nearly equal they may be denoted by a and $a + \delta$, to indicate the fact that their difference is a small magnitude. Then

$$\sqrt{a(a + \delta)} = \sqrt{a \cdot a(1 + \delta/a)} = a\sqrt{1 + \delta/a} = a(1 + \delta/2a)$$

since δ/a is also a small magnitude. But

$$a(1 + \delta/2a) = a\,(2a + \delta)/2a = \frac{(a) + (a + \delta)}{2}$$

In other words, if two quantities are nearly equal the square root of their product (or *geometrical mean*) can be found by taking their average (or *arithmetical mean*).

If an object has an apparent weight of m_1, when

placed on one pan of a balance, and m_2 when on the other pan, it can be proved that its true mass is $\sqrt{m_1 m_2}$.

Weigh your whole set (16 oz.) of avoirdupois weights, considered as an unknown mass, on the platform balance against the brass metric weights. Repeat the process on the other pan of the balance. Find the true mass of the avoirdupois set in grams from the formula:

$$\sqrt{a(a + \delta)} = \frac{(a) + (a + \delta)}{2}.$$

Draw a diagram similar to the one on page 30, but with a very small angle at O. It will be almost self-evident that

$$tan\ \delta = \delta$$

and

$$sin\ \delta = \delta,$$

remembering that the angle δ is to be measured as the ratio of arc to radius.

By consulting the table of circular functions on page 141, find the largest whole number of degrees for which $tan\ \delta = \delta$ to four decimal places. The numerical measure of each angle will be found beside the number of degrees in the column headed RAD, an abbreviation of *radian*, the name sometimes given to the unit of angle. For how large an angle is it true that $sin\ \delta = \delta$ as far as four decimal places?

If an accuracy of three decimal places is all that is needed how large can δ be without differing from $tan\ \delta$ and $sin\ \delta$ respectively?

Recapitulation.—The formulæ for deltas are collected for reference here. Notice that the second and third are special cases of the first, and the first formula

for n equal to a whole number only, is a special case of the fifth.

$$(1 + \delta)^n = 1 + n\delta$$
$$1/(1 + \delta) = 1 - \delta$$
$$\frac{1 + \delta_1}{1 + \delta_2} = 1 + \delta_1 - \delta_2$$
$$\sqrt{1 + \delta} = 1 + \tfrac{1}{2}\,\delta$$
$$(1 + \delta_1)\,(1 + \delta_2)\,(1 + \delta_3)\ldots = 1 + \delta_1 + \delta_2 + \delta_3 \ldots$$
$$\sqrt{a(a + \delta)} = \frac{(a) + (a + \delta)}{2} = a + \delta/2$$
$$\tan\delta = \sin\delta = \delta$$

Find $\sqrt{50}$ by using the δ-formulæ.

Answer: $\sqrt{50} = \sqrt{49 - 1} = \sqrt{49\,(1 - 1/49)} = 7\sqrt{1 - 1/49} = 7\,(1 + 1/98) = 7\left(1 + \frac{1}{100-2}\right) = 7\left(1 + \frac{1}{100}\,(1 + .02)\right) = 7\,(1.0102) = 7.0714.$

The first four figures of this result are correct. Extreme accuracy cannot be expected where a delta is larger than .005; in most physical calculations it is much smaller than this.

Find $400 \div 797$. Suggestion: divide through by 800 in order to put the divisor in the form $1 + \delta$.

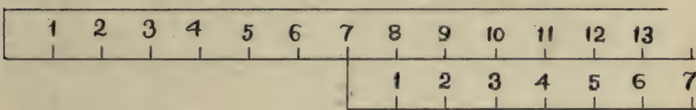
Find $.504 \times .498$.

VI. THE SLIDE RULE

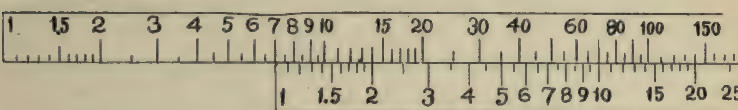
Apparatus.—A ten-inch slide rule, with A, B, C, D, L, S, and T scales, a runner, and a list of metric equivalents.

Addition with Two Scales.—If two scales of centimetres are laid parallel so that the zero of the second one coincides with the seventh division of the first

it is evident that the fifth division of the second will be opposite the twelfth division of the first. A length of 5 has been added to a length of 7, and the result is seen at a glance to be 12. It will also be noticed that the arithmetical difference between scale divisions that lie opposite each other is everywhere the same and is equal to the number on the first scale which is opposite the beginning of the second.



Multiplication with Logarithmic Scales.—If the same experiment is tried with two scales whose divisions are not a succession of whole numbers but the logarithms of such a series the result will be different, for adding logarithms corresponds to multiplying natural numbers. Accordingly if we start at *log* 7 and measure a further distance of *log* 5 we shall come out with



log 7 + *log* 5, which is not equal to *log* (7 + 5) but to *log* (7 × 5). Notice that not only does five on the second scale come opposite 35 on the first, but also that quotients of corresponding numbers are everywhere equal to 7, just as differences are in the first diagram; and furthermore, the upper scale in the second diagram forms a multiplication table (a 7 table

in this case) for the numbers on the lower scale just as it did an addition table in the other diagram.

The Slide Rule.—The apparatus called a slide rule is essentially a ruler containing a groove in which is a movable slide. Logarithmic scales are so marked that one of them (the "A scale") can be held stationary while another (the "B scale") can be placed in any required position below it.

Two scales (C and D) along the lower edge of the slide can be used in the same way and can be read a little more accurately on account of their subdivisions being larger, but the two upper scales (A and B) will be found the most convenient for general use. Each one of them is really two complete logarithmic scales from 1 to 10, but the right-hand half is often marked 10 to 100. If the numbers are the same on both halves it is necessary to remember that any graduation, such as 7, can be used to represent 7, or 70, or 7000, or .07, as may be required. Care should be taken to avoid mistakes in reading the graduations between the whole numbers, for the smallest interval is not everywhere the same, being .1, .05, .02, and .01 in the various parts of the scales. Directions have been worked out for finding the position of the decimal point in a result obtained by means

of the slide rule, but it is preferable to form the habit of always determining the approximate answer before making any arithmetical calculation, whether it is made with the slide rule or not.

Checking by Approximation.—This is one of the best methods of checking the accuracy of the result and guarding against ridiculous mistakes. For example, suppose 23.4 is to be multiplied by 82.9; the computer should say that $20 \times 80 = 1600$, $\therefore 23.4 \times 82.9$ must be somewhat larger than 1600. When the answer is found by the slide rule to have the significant figures 194 there can be no doubt that it should be pointed off 1940, not 19.4 or 19400. Furthermore if the answer obtained either with the slide rule or by abridged multiplication or by any other method should come out 613 it would be evident that a mistake had occurred somewhere.

Multiplication.—To find the product of two numbers with the slide rule either end of the B scale should be set opposite one factor on the A scale, then the other factor on the B scale will be opposite the product on the A scale. Try this process with small numbers by setting 1 on B opposite 3 on A and observing that the position of 2 on B gives 2×3 on A.

Multiply 2×4; 2×5; 2×6; 3×9; 7×8; 7×13; 7×16.

Remember that the two halves of each scale are identical. If 16 (or 1.6) when taken on the right half of the B scale falls beyond the end of the A scale the same result can be read over 16 (or 1.6) on the left half.

Multiply 1.5 by 2; 1.8 by 2; 1.7 by 2.1 (estimate third figure of product); 17.8 by 2; 1.79 by 2.53.

Find $0.6 \times 183.$; 7.3×1.09; 0.073×0.0016; $325. \times 106.5$.

The use of the "standard form" often makes the preliminary checking easier; thus in the last two examples given, $(7+) \times 10^{-2}$ multiplied by $(2-) \times 10^{-3}$ gives 14×10^{-5}, or .00014; and 3×10^2 multiplied by 1×10^2 gives 3×10^4 or 30000.

Division.—In the first diagram, above, it may be considered that a length of 12 is laid off from left to right, and then, beginning at the point 12, a length of 5 is laid off to the left, or subtracted from the original 12, giving a final result of 7. Similarly in the second diagram *log* 35 minus *log* 5 equals *log* 7. The rule for division is accordingly: place the divisor on B under the dividend on A and read the answer on A over either end of the B scale.

Divide 35 by 5; 30 by 5; 30 by 4; 11 by 4; 11 by 7; 11 by 10; 11 by 11; 11 by 12; 11.8 by 99.; 114. by 3.4.

Ratio and Proportion.—In the first diagram $8 - 1 = 9 - 2 = 10 - 3 = 7$. Since subtraction of logarithms corresponds to division of natural numbers the second diagram shows $14/2 = 28/3 = 35/5$, etc. That is, with the slide rule set in a given position any two opposite numbers are in the same ratio as any other two. Set 6 under 2 and notice that 15 is under 5. Solve the following proportions by setting the rule so that the answer is always found on the A scale.

$$6 : 2 :: 15 : x;$$
$$3 : 2 :: 9 : x :: 12 : y :: 10 : z;$$
$$31 : 750 :: .005 : x$$

(First say 750 is about twenty times as large as 31.)

Solve the following equation as accurately as possible:

$$26 : 66 :: 1 : x$$

If 26 inches = 66 centimetres what is the length of one inch? What is the *approximate* length of 4 inches? How many centimetres in 41 inches? Turn to the back of the slide rule and see what statement you can find of the relation between centimetres and inches. Look for a similar statement in regard to pounds and kilograms and calculate your own weight in kilograms, remembering to set the slide rule so that the answer is always found on the A scale.

Reciprocals.—Set any number on B under 1 on A; what do you read on A over 1 (or 10, or 100) on B? Set any number on C over 1 or 10 on D; what do you read on D under 1 or 10 on C? What advantage have the C and D scales as compared with the A and B scales?

Try multiplication, division and proportion on the C and D scales, remembering to set the slide so that the answer always comes on the stationary (D) scale. In multiplication if the answer runs off the end of the rule re-set the slide with 10 instead of 1 on C opposite the first factor, and in division read the result under 10 instead of under 1 when necessary. If the fourth term of a proportion cannot be read the end of the C scale which is over the D scale must be precisely replaced by its other end.

Set 26 over 66 and find the number of centimetres in five inches. After setting the slide move the transparent runner so that its vertical black line exactly covers 1 on C, and without allowing its position to shift set 10 on C directly under it. Then under 5 on C read the answer on D.

Squares and Square Roots.—Set the slide so that 1 on C is just opposite 1 on D, and move the transparent runner so that its vertical line falls on 4 of the A scale. Where does it cut C and D? Set it successively at 9, 16, 25, 36, and 49 on A, and read D. Set it at 9, 8, and 7 on D and read A.

The square of any number is given unequivocally (except for the location of the decimal point, which is easily determined by inspection) on A above the number itself on D. Care, however, is necessary in reversing the process, for a given arrangement of significant figures has two different square roots, as is shown in the following equations:

$\sqrt{1500} = 40 -$; $\sqrt{150} = 12 +$; $\sqrt{15} = 4 -$; $\sqrt{1.5} = 1.2 +$; $\sqrt{.15} = .4 -$; $\sqrt{.015} = .12 +$; $\sqrt{.0015} = .04 -$; $\sqrt{.00015} = .012 +$.

One of these, 121, will be found under the 150 of the left half of the A scale, the other, 383, under the 150 of the right half. The simple precaution of making a rough mental preliminary calculation of the root will avoid the possibility of using the wrong number. For example, is the square root of .036 given by the significant figures 19 or 60, and how should it be pointed off?

There are several slightly more complicated forms of problem which can be solved with a single setting of the slide rule. Thus using the upper scales we can not only read the value of a/b by setting b under a, but we can also use the method of slide-rule multiplication to give the value of $\frac{a}{b} \times c$ without re-setting the slide, and without even stopping to determine the

numerical value of a/b. Similarly a^2b will be found on A opposite b on the B scale if 1 on C is set to a on D; and $\sqrt{ab}$ will be found on D under b on B if 1 on B is set to a on A. A series of fractions like a/m, b/m, c/m, d/m, ... can be read off by merely setting the slide rule for $1/m$ and looking opposite a, b, c, d, ... Most slide rules have two "cylinder points" marked off on the C scale at $\sqrt{(4/\pi)}$ and $\sqrt{(40/\pi)}$; by placing either one of these opposite the diameter of a cylinder the length of the cylinder on B will indicate its volume on A.

Determination of Circular Functions.—By removing the slide, turning it over, and replacing it so that the ends of the S and T scales coincide with the ends of the A and D scales respectively, the sine of any angle will be found on A opposite the number of degrees on S; and the tangent will be found on D opposite the number of degrees on T. The decimal point is located by recalling the facts that $\sin 90° = 1$ and $\tan 45° = 1$.

By drawing a right-angled triangle the student will see that $\tan\ (45 + a)$ is the reciprocal of $\tan\ (45 - a)$, and this fact is made use of if tangents of the larger angles are to be obtained from the slide rule. For sines less than .01 and tangents less than .1 different types of slide rule employ different methods, usually based upon the formulæ $\tan\ \delta = \sin\ \delta = \delta$.

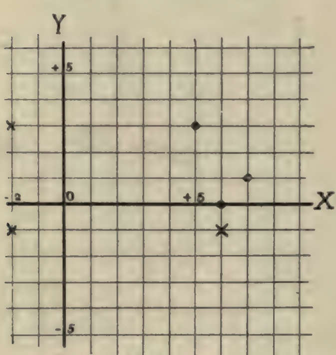
The scale between S and T is generally marked L and is used for obtaining logarithms. Set 1 on C opposite n on D and $\log\ n$ will be found on L opposite a special mark on the back of the slide rule.

VII. GRAPHIC REPRESENTATION

The position of any point on the surface of the earth may be indicated by two numbers: one, the longitude of the point, expresses its distance to the east or west of an arbitrary line, the meridian of reference; the other, its latitude, gives its distance north or south of another line.

In almost all branches of science the opposite of this process has been extensively used, and the location of a point on a diagram is made to represent the numerical values of two quantities that are in any way related to each other.

Graphic Diagrams.—If two straight lines are drawn, one horizontal and one vertical, the position of any point in the same plane can be represented by two numbers, one giving its distance to the right of the vertical line, the other its distance above the horizontal line; and conversely any two numbers can be represented by properly locating the point.



Paper that has been ruled in small squares is convenient, although not necessary, for constructing what are known as *graphic diagrams*. A convenient horizontal line is chosen for one axis of reference and is called the x-axis, and a vertical line of reference is

called the y-axis. To represent the numbers 5 and 3 a point is located 5 units to the right of the y-axis and 3 units above the x-axis, and is spoken of as the point (5,3). The above diagram shows this point and also the points (7,1) and (6,0) by small dots; and by going to the left and below the axes it is possible to represent negative as well as positive values, as is seen by the small crosses located at $(+6, -1)$ $(-2, +3)$, and $(-2, -1)$.

The following table contains different values of x and the corresponding values of y. Pick out the values of both x and y that are algebraically largest and smallest and mark the axes of reference in your notebook in such a position that there will be space enough

x	y	x	y	x	y
1	1	9	− 1	1	− 7, − 9
2	2	10	0	−1	− 1, − 5
1	3	11	− 1	2	− 10
2	4	12	− 2	5	− 11
3	5	13	− 3	2	− 8
4	6	12	− 4	3	− 11
5	5	11	− 5	−2	0
6	4	10	− 6	−3	− 1, − 5
7	3	9	− 5	−4	− 2, − 4
6	2	8	− 6	0	0, − 6
7	1	7	− 7, − 9	−5	− 3
8	0	6	− 8, − 10.	−2	− 6

for the points representing the extreme values. Then take each pair of values in succession and plot each corresponding point.

Graphic diagrams are used chiefly to show the relation between two variable quantities. A quantity that is varied arbitrarily is usually represented by x, or hori-

zontally, while the dependent variable or related effect is represented by y or vertically. Thus, if a gas is compressed it will have a certain volume for each amount of pressure, and its condition at any time can be represented by plotting vertically the resultant volume due to the independently varied pressure that is represented horizontally.

The following table is an example of the typical fluctuations in body-temperature of a normal individual. Plot the corresponding points on a graphic diagram in which each small square represents one hour horizontally and one-tenth of a degree vertically. In order to save space allow only enough vertical distance for the plotted points, namely 13 squares, and lay off a numbered scale on the y-axis to show that the x-axis or zero of temperature must be considered as situated far below the diagram. Note how much more "graphic" the diagram is than the table; how it shows at a glance facts that could be gleaned from the table only with much greater effort.

hour	*temp.*	
	A.M.	P.M.
12	36.9	37.4
1	36.8	37.4
2	36.8	37.6
3	36.6	37.5
4	36.4	37.5
5	36.5	37.6
6	36.6	37.6
7	36.8	37.7
8	36.9	37.6
9	37.1	37.4
10	37.2	37.4
11	37.2	37.2
12	37.4	36.9

Make another diagram which is an exact duplicate of this one. In one diagram draw a straight line from each point to the next one so that all the points are united by a broken line. In the other try to draw a smooth curve which shall follow the general trend of the points as nearly as possible and lie above as many points, approximately, as it lies below. It should

show not more than one downward swing and one larger upward swing.

The method of connecting points with a broken line is used when it is desired to represent the separate tabular values without implying anything about intermediate values, as in most cases where the law of variation is unknown. The method of drawing an approximate curve is known as "smoothing" the graphic diagram; it is used only when the discrepancies between the actual points and the smoothed curve are no greater than can be accounted for by unavoidable errors in obtaining the numerical values, or when they are known to arise from unimportant causes. A third method is possible when the quantities vary according to some mathematical law. In such case a smooth curve can generally be so drawn as to pass exactly through all of the plotted points, as will be seen in the examples given later; and if an equation is to be represented graphically only enough points need be plotted to allow the shape of such a curve to be satisfactorily determined.

Graph of an Equation.—A single equation in two unknown quantities has an infinite number of solutions. Any value may be assigned to one unknown quantity and the corresponding value calculated for the other. A single solution can be represented by a point on a graphic diagram and the totality of such points form a curved or straight line, which is called the "curve" of the equation. Thus if $y = 2x + 3$ the values of x and y given in the table are all solutions and may

x	y
0	3
1	5
2	7
3	9
4	11
−1	1
−2	−1
−3	−3
−4	−5

all be plotted on a graphic diagram. The "curve" of this equation is easily seen to be a sloping straight line, and a trial will show that fractional values of x or y will also give points that lie exactly on the line.

In plotting an equation the table of values should always be first calculated, substituting successive positive and negative integral values of x (and fractional values if necessary) and solving for y, and then the available space on the paper considered and the axes drawn in a suitable location. If the x-values, or "abscissæ" as they are called, are either very large or very small in comparison with the y-values or "ordinates" the two quantities may be plotted with scales of different size; but along each axis, considered by itself, equal distances must always correspond to equal numerical differences, and x-values must always increase to the right and y-values increase upward. It is often advisable, in cases where the curve will have a fairly uniform slope and the relative values of the x-unit and the y-unit are unimportant, as in the extrapolation diagram, page 77, and in the greater part of the diagram on page 132, to choose such scales that this slope will be approximately 45°. In the following exercises of this lesson, however, a single square of the paper is to represent one unit, in each direction, except where otherwise specified.

On the squared paper of the note-book lay off a rectangle 10 squares wide and 20 squares high. Draw the axes so the rectangle extends from $x = -5$ to $x = +5$ and from $y = -10$ to $y = +10$. Within the limits of this rectangle plot the curve of the equation $y = 2x$ by assigning integral values to x from

-5 to $+5$, and discarding any points whose y values do not come within ± 10. On the same diagram plot $y = \frac{1}{3}x$, $y = -\frac{1}{2}x$, and $y = 2x + 4$. How many of these are straight lines? On another diagram, between the limits $y = -5$ and $+25$, and $x = \pm 10$, plot the following: $y = x^2$; $y = x^2/2$; $y = x^2/10$; $y = -x^2/10$; $y = 4$; $x = 3$ (hint: substitute various values of y and calculate the corresponding values of x).

All curves of the form $y = ax^2$ are similar figures; $y = x^2$ and $y = \frac{1}{10}x^2$ differ in size but the portion of the latter extending from $x = -5$ to $x = +5$ is of exactly the same shape as the part of the former that is included between $x = -1$ and $x = +1$. The curve is called a parabola.

Plot $y = x^3$ using the scales that you think best. Carry the x-values only far enough to make the y-values increase or decrease rapidly.

Plot $y = 30/x$ with enough points to show clearly the form of both branches of the curve; plot on the same diagram $y = 1/x$.

Draw the curve of $y = 1/x^2$. How would the curve of $y = 30/x^2$ differ from it. How would $y = -1/x^2$ compare with it?

Plot $y = \log x$ from $x = 0$ to $x = 10$ using a large scale on the x-axis (for example, five squares for each unit), and one about twice as large on the y-axis. Locate as many points as are needed to determine a smooth curve; the whole numbers from 1 to 10 ought to be sufficient.

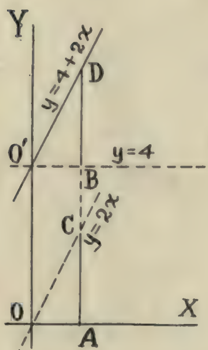
Turn back to the page left vacant in the lesson on logarithms, and plot carefully the equation $y = e^{-x^2}$ from $x = -0.5$ to $x = 2.5$ or 3. Turn the note-book

so that the x-axis will lie lengthwise of the page and use as large a scale as possible (say 20 squares = 1.0). If y equals $+0.7$ when $x = +0.6$ what does y equal when $x = -0.6$? Note what value of x is required in order to make the corresponding ordinate indistinguishable from zero on your diagram. The importance of this curve will be seen later when taking up the subject of errors of observation.

VIII. GRAPHIC ANALYSIS

Apparatus.—Fine black silk thread; slide rule.

The line plotted on a previous diagram for $y = 2x + 4$ will be seen to have each of its points four units higher than a corresponding point on the line $y = 2x$; in fact this could have been inferred without plotting either of the lines, for it is evident that whatever may be the value of x the y of the former equation is greater than that of the latter by four units.



In connection with the study of physical changes it is more convenient to put such an equation in the form $y = 4 + 2x$ and to consider the y-values of the curve as the sum of the constant 4 (AB in the diagram) and $2x$ (AC in the diagram). BD is made equal to AC so that the new y-value, or AD, is AB + AC, or $4 + 2x$.

The plot of $\mathbf{y = a + bx}$.—Accordingly, to plot such a curve as $y = a + bx$, whatever values a and b may

have, it is necessary only to mark a new zero-point, or "origin" of the graphic diagram, as it is called, at a distance a above the old origin, and to plot $y = bx$ from this new starting-point. Notice that b is the numerical tangent of the slope of the line, and a is the "intercept" on the y-axis, or the distance above O to the point where the y-axis is cut by the curve.

Draw the following equations on one diagram by laying off the y-intercept (OO′ in the diagram) and drawing a straight line that has the proper slope (ratio of BD to O′B in the diagram): $y = 2 + 3x$; $y = 1 + 0.5x$; $y = -2 + 0.5x$; $y = 2 + 0.5x$; $y = 2 + 0.25x$; $y = 2 - 0.25x$; $y = 2 - 0.5x$.

If there is any doubt as to the meaning of a negative slope a few points should be plotted from the equation in the usual manner.

It should be noticed that the form $y = a + bx$ cannot be used for every straight line that can be drawn. The general equation of the straight line is $Ax + By + C = 0$. This can be made to represent lines parallel to the y-axis by putting B equal to zero; such lines, however, are never needed for expressing physical changes.

If two quantities are always proportional or vary directly with one another, as in the case of the absolute temperature of a gas and its resultant pressure, what kind of a graph would be obtained by plotting one of them as x and the other as y?

If equal changes in one quantity always correspond to equal changes in another what kind of a graph expresses their relationship? Explain just what the difference is between this graph and the previous one.

An example of this kind of relationship is given by a metal bar which has a certain length at a certain temperature and undergoes changes in length which are accurately proportional to the changes in its temperature. If there is any trouble in answering the question an arbitrary point, say the point, (a, b), should be plotted, and then two or three other points, (x_1, y_1), etc., which are so situated that $(x_1 - a)/(y_1 - b) = (x_2 - a)/(y_2 - b) = (x_3 - a)/(y_3 - b)$, etc.

The "Black Thread" Method.—If a set of experimental measurements, such as those of the temperature and length of a metal bar, are found to correspond approximately to the "straight line law" they may be plotted and "smoothed" by drawing the straight line that comes closest to all the points. This is called the "black thread" method because the position of the line is decided upon with the aid of a stretched thread instead of a ruler; the thread and the points can all be seen at the same time, while a ruler would hide half of the points if it was so placed as to lie evenly among them.

x	y
1	9.8
2	8.5
3	8.0
4	7.2
5	6.7
6	6.5
7	6.2
8	5.5
9	5.0
10	4.1
11	3.9
12	3.2
13	2.3

Plot these tabular values as accurately as possible, marking each point by a minute dot surrounded by a small circle, or by a cross composed of a short vertical line to mark the exact value of the abscissa and a short horizontal line at the exact height of the ordinate.

See that the page of the note-book rests in a perfectly flat position and stretch a fine black silk thread in a straight line that follows the general direction of the points. Move it a trifle upward and downward

along the y-axis, also rotate it slightly, both clockwise and counter-clockwise. When you finally get the position that you think lies most closely and evenly among the points notice exactly where the thread cuts both the x-axis and the y-axis, and from these two values calculate its slope, noticing whether the value is positive or negative. What is the equation of the line indicated by the thread, expressed in the form $y = a + bx$?

What are the distances intercepted on the two axes by the line $x/a + y/b = 1$? How can you use this form to make it easier to calculate the equation of a black thread determination?

The Plot of $y = a + bx + cx^2$.—Just as the curve $y = a + bx$ can be considered as having its ordinate for each value of x built up of the ordinate a plus the ordinate bx, so the more general form $y = a + bx + cx^2$ can be considered as made up of the straight line $y = a + bx$, on which is superimposed the parabola $y = cx^2$. Curiously enough, this also represents a parabola in every case where c is different from zero.

On a single graphic diagram plot $y = -0.1x^2$ and $y = 3 + 0.5x$. Then add the ordinates of the former algebraically to the latter and draw as smoothly as possible the resultant curve $y = 3 + 0.5x - 0.1x^2$.

If an experimental curve looks as if it could be represented by $y = a + bx + cx^2$ it is possible to draw an approximate tangent, find its equation, and then determine the coefficient of x^2, but it is usually more satisfactory to proceed as in the following example, completing a free-hand parabola as far as the point where it runs horizontally if such a point is not already present.

The density of water at different temperatures is given in the table. If the density is called y and the temperature x what are the numerical values of the coefficients in the equation $y = a + bx + cx^2$?

temp.	*dens.*
0	.99984
2	.99994
4	.99997
6	.99994
8	.99984
10	.99970
12	.99950
14	.99924
16	.99894
18	.99859
20	.99820
22	.99777
24	.99729
26	.99678
28	.99623
30	.99564

x	y	$\sqrt{y}$
2	3	
4	13	
6	27	
8	47	
10	73	
12	103	
14	138	
16	177	
18	220	
20	268	
22	319	
24	374	
26	433	

The first step is to make a careful graph. The scale on the y-axis needs to extend only through the numbers .995, .996, .997, .998, .999, 1.000. It is apparent that the curve is very much like a parabola, and if we should arbitrarily put the origin of the graphic diagram at the point (4., .99997) the equation would be $y = -mx^2$. The positive values of its negative ordinates will be obtained by subtracting each tabular density from .99997, and the new values of x will be 4 less than before. The results are shown in the next table, with their decimal points omitted. Now, with

your slide rule, read off the square roots of the numbers in column y and enter them in the column $\sqrt{y}$. If y is proportional to x^2, as shown by the equation $y = -mx^2$, then the square root of y must be proportional to x itself, and their relative magnitudes can be determined by the black thread method.

Plot the corresponding values of x and $\sqrt{y}$ and determine the slope with the black thread, remembering that the line must pass through the origin. Your value of the tangent should not differ greatly from $\frac{5}{6}$ or 0.83.

From $\sqrt{y} = .83x$ it follows that $y = -.69x^2$, y being expressed in hundred-thousandths, and with the aid of the slide rule the values of $.69x^2$ are read off as shown in the table. In the last column they have been increased by 3 to make it more convenient to plot them.

On the same graphic diagram plot the values of $y + 3$ in .00001's downward from the line $y = 1.00000$, remembering that $x = 0$ is now at the apex of the curve, and notice how closely the graph of the equation coincides with the graph of the experimental values.

x	$.69\,x^2$	$y + 3$
0	0	3
2	3	6
4	11	14
6	25	28
8	44	47
10	69	72
12	99	102
14	135	138
16	177	180
18	224	227
20	276	279
22	334	337
24	398	401
26	467	470

To reduce the equation $100000y = -.69x^2$ to the original axes notice that any point whose abscissa and ordinate are $(x,\ y)$ with reference to the new axes will have an abscissa of $x + 4$ (call it x') and an ordinate of $y + .99997$ $(= y')$ when referred to the old axes. Hence $x = x' - 4$ and $y = y' - .99997$; substituting these values in

$100000y = -.69x^2$ we obtain $100000\,(y' - .99997) = -69\,(x' - 4)^2$, or, dropping the accents and simplifying:

$$y = .99986 + .0000552\,x - .0000069\,x^2.$$

Writing d for the density of water and t for centigrade temperature this becomes

$$d = .99986 + .0000552\,t - .0000069\,t^2.$$

Linear Relationship by Change of Variables.—Any law of change can be expressed by the formula $y = a + bx + cx^2 + dx^3 + \ldots$ if enough terms are used, but the method soon becomes difficult to handle. Sometimes the appearance of the curve makes it possible to guess that its equation is of some particular form, such as

$$y = a/x, \text{ or } y = e^{ax}, \text{ or } y = \frac{ax}{b + x},$$

or the form may be known from theory. In such cases it is best to make use of a procedure similar to that employed for determining m in the equation above, $y = -mx^2$. For example $y = \dfrac{ax}{b + x}$ will give a curved line, but by calculating y/x it will be found that for this particular equation the graph of y and y/x will be a straight line.

The volume of a certain mass of air was found to vary, with the pressure to which it was subjected, according to the numbers in the following table. Represent the relation between volume and pressure by an equation.

Here it is known from Boyle's law that the pressure and volume are inversely proportional, or $v = a/p$. If the pressure is denoted by x and the volume by y a curve can be plotted, but a straight line ought to be obtained by plotting y and $1/x$.

Choose the x- and y-scales so that the line is neither very steep nor very flat, use the black thread method, and determine the value of a in the above equation.

P (*atmo.*)	V (cm^3)
1	10
1.5	6
2	5
2.5	4
3	3
3.5	3
.75	13
.50	19

Graphic Interpolation.—In the case of most experimental curves and mathematical equations the fact that the positions of certain points are known makes it possible to draw a smooth curve through them and thus determine the location of any intermediate point.

Turn back to the graph of $y = log\ x$. How high is the curve above the base-line for $x = 3.5$? Do not draw the ordinate but measure it accurately. In the same way measure the logarithms of 2.5, 1.2, and 0.8. When the independent variable in this equation is less than 1 the ordinate should be measured downward past the curve to the next (negative) whole number, and then another measurement made from this point upward to the curve. This will put a number like -3.8 in the form of minus four plus the other two-tenths, or $\bar{4}.2$, as a logarithm is usually written.

Join the two points of the curve (2, *log* 2) and (3, *log* 3) by a straight line on your diagram. In the lesson on logarithms it was assumed that the value of *log* 3142 was one-fifth of the way from *log* 3140 to *log* 3152 because the natural number 3142 was one-fifth of the

way from 3140 to 3150. If this assumption were absolutely accurate what would be the shape of the logarithmic curve between $x = 3140$ and $x = 3150$? Can you make the same assumption for the stretch of curve lying between (2, .301) and (3, .477)? Compare the ordinate of the middle point of the chord just drawn with the interpolated value of *log* 2.5 and with the value of *log* 2.5 obtained from the tables. Why is the first assumption justifiable in using a table of logarithms?

Turn to the 5-place table on page 144 and notice that the slope of $y = log\ x$ varies so rapidly between $x = 1.000$ and $x = 1.100$ that the "argument" (or *natural number*, or *antilogarithm*) has to be given at intervals of .001 in order to make the process of interpolation trustworthy to 5 decimal places. Beyond $x = 1.100$, however, intervals as large as .01 can be used without the chord deviating from the curve enough to affect the fifth decimal place.

What logarithm is accurately represented by the ordinate of the middle of your chord, and which of the formulæ for approximate calculation with small magnitudes is illustrated by this diagram?

Plot the density of water on a large scale, allowing the height of a small square to represent .00001 and using five squares horizontally for 2 degrees. Mark down the points 0°, 2°, 4°, 6°, 8°, and draw a smooth free-hand curve through them; unite them also with chords, and find the density at 1° and 3° or at 5° and 7° by both methods of interpolation. For which points is the arithmetical method liable to be erroneous and for what values could it probably be trusted?

Graphic interpolation is especially useful in cases of smoothed curves. If two successive ordinates happened to have errors of the same sign the value obtained by arithmetical interpolation would be less accurate than that obtained graphically.

From your graphic diagram (see page 64) find the most probable value of the average temperature of a healthy individual at 9:30 A.M.

Extrapolation.—The process of finding the value of a function beyond its known values instead of between them is called extrapolation. It needs to be employed with extreme caution where the general law of change is unknown.

Draw a graphic diagram of the population of California at ten-year intervals as given in the table, connecting the points with a smooth curve. Continue the curve so as to determine the value for 1910.

year.	*pop'l'n.*
1850	93,000
1860	380,000
1870	560,000
1880	865,000
1890	1208,000
1900	1485,000
1910	

Reconstruct the curve, if necessary, so as to give 2.38×10^6 for 1910, and find what the population will be in 1920.

IX. THE PRINCIPLE OF COINCIDENCE

Apparatus.—Two metre sticks; card-board; knife; square wooden block; slide rule.

Measure one inch in centimetres, millimetres, and estimated tenths of a millimetre. Measure a length of two inches in the same way, and divide the result by two. Take one-tenth of the measured length of

ten inches and compare with the two other results. Notice that a large quantity can usually be measured more accurately than a small one.

Measurement by Estimation.—In order to make a more accurate determination of the length of an inch lay one metre stick on the table with the metric graduations upward and place the other beside it with the graduation in inches and tenths of an inch upward. See that the scales are in close contact and note down the number of centimetres and hundredths indicated on one scale by the mark 1 inch on the other. Note also the indication of the mark 31 inches. Move one scale along the other at random and repeat the observations until ten sets have been taken. The distance measured should always be the same, 30 inches, but it should be measured at other places, such as the mark 4 and the mark 34 Never use the mark 0 at the very end of the stick, as it is often inaccurate on account of wear.

Tabulate the results in a form like the following:

	1st line	2d line	Interval	
Cm scale	20.05	45.50	25.45	
Inch scale	1.00	11.00		10.00
Cm	50.00	75.47	25.47	
Inch	2.00	12.00		10.00
Cm	10.06	35.46	25.40	
Inch	1.50	10.50		10.00
Sum			254.02	100.00
Average . . .			25.402	10.000

Length of an inch: — 2.5402 cm.

Measurement by Coincidence.—Set the two scales so that any whole number of inches near one end of one metre stick is exactly opposite some whole number of centimetres on the other. Hold the two sticks firmly together and look again to see that the coincidence is exact. Find another place, at least 30 cm distant from this point and preferably further, where there is another exact coincidence, this time between any centimetre or millimetre graduation and any graduation of inches or tenths. Try to decide whether the coincidence is exact or whether the imaginary central axis of one line lies a little beyond the other, and if the coincidence is faulty choose a better one elsewhere.

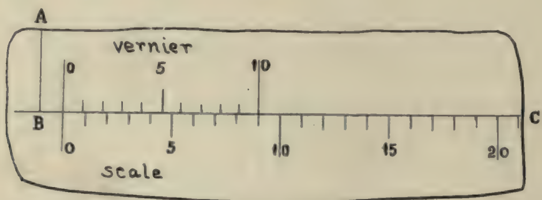
Record the results as in the specimen table, re-set the two scales, and repeat until 10 determinations have been made.

	1st line	2d line	interval	cm in 1 in.
Cm In.	90.000 4.000	35.400 25.500	54.600 21.500	2.5395
Cm In.	15.000 6.000	71.400 28.200	56.400 22.200	2.5405
		Average . .		2.5401

It is not absolutely accurate to assume that the coincidences are exact to a hundredth of the smallest graduation of the scale, but they certainly ought never to have an error of as much as half a tenth, or .05, and rarely as much as .01 if the work is carefully done.

The Vernier.—Rule a straight line along a strip of cardboard that measures about 10 × 25 cm. Lay off

a scale of centimetres below it and a scale of ten multiples of 9 mm. above it as shown in the diagram. Cut the cardboard from A to B and from B to C, so that the short scale can slide along the centimetre scale. The small scale is called a vernier and is used to indicate tenths of the divisions on the large scale. Move it to the right very slowly, making its first division coincide with a division on the main scale, then further until the second division does likewise, and so on, until the tenth vernier division is opposite some division of the other scale. Notice that the zero of the vernier has been moved just one centimetre during the process; hence if some other division of the vernier, such as No. 3, coincides with *any* centimetre division the zero of the vernier must be three-tenths of an interval beyond some scale mark.



Separate the jaws of this model vernier caliper by a distance of three and a half centimetres, as nearly as you can by estimation; then look at the vernier and notice that its zero is beyond the mark 3 of the centimetre scale and that the division 5 of the vernier coincides with some division (no matter what) of the main scale. The reading is accordingly 3.5 cm. In the same way read the indicated length when the caliper is set at random.

Hold the two parts of the caliper in alignment by pinching the line BC between the fore-finger and thumb of both hands and use the apparatus to make twelve measurements of the *thickness* of the wooden block at different positions around its edge. Find the average thickness of the block, carrying the calculation out to one more decimal place than the individual measurements.

Slide-Rule Ratios.—Set the slide rule so that the number seven on the B scale is under 22 on A, and see if the end of B comes opposite the special mark π on the A scale. If not, set 1 on B accurately under π on A and look along the scales until two graduations are found which are precisely opposite. See if these two numbers are multiples of 22 and 70, and if so try to find another ratio which is indistinguishable from π and is not identical with 22/7.

Numbers like the 22 and 66 on the back of the slide rule for the ratio of inches and centimetres are so chosen as to give the correct value to at least as many significant figures as can be read with the apparatus used. Some slide rules give the equivalent as 50 in. = 127 cm. This has the disadvantage of not being quite as easy to set on the A and B scales as on C and D, but for use with a very finely graduated instrument or with one on which the scales are 20 inches long instead of 10 it has the advantage of greater accuracy, for $66.0000 \div 26.0000 = 25.3846$ while $127.0000 \div 50.00000 = 25.40000$. The former ratio is correct to three significant figures only, while the latter is correct to five.

The number of avoirdupois or troy grains in one gram is 15.432. Find a slide-rule ratio which gives this value correct to four significant figures.

X. MEASUREMENTS AND ERRORS

Apparatus.—A vernier caliper reading to tenths of a millimetre; 100 variates (such as the seeds of *Phaseolus*) for measurement.

Direct and Indirect Measurements.—Measurements are classified as *direct* and *indirect*. The ordinary processes of measuring length or weight are direct, because the unknown length is placed beside a standard series of multiples and fractions of the unit of length and is directly compared with it, and an unknown weight is directly balanced against a series of known weights until its exact equivalent is determined. An example of an indirect measurement is the usual method of determining density. The volume of an object and its mass are determined directly, and its density is then obtained from the direct measurements by calculating the ratio of mass to volume.

Independent, Dependent, and Conditioned Measurements.—Measurements are also classified as *independent*, *dependent*, and *conditioned*.

If there is a theoretical relationship that must always hold good between different quantities, their measurements are said to be conditioned. An example of this is the measurement of the three angles of any triangle. Their sum must necessarily be 180°, or π.

Measurements are said to be dependent if one measurement is allowed to influence or bias the ob-

server when making a later measurement of the same quantity. With delicate measurements this effect is so hard to avoid, even if the observer has the best of intentions, that it is always advisable to guard against it by some such device as hiding the scale of an apparatus from view until after the indicating mark has been set in the position that has to be read. Measurements are also dependent if any essential step in making them is not repeated in successive determinations but is assumed to have its effect remain unchanged during the series. Thus, in making independent measurements by the method of coincidences the scales of length were not held in one position while several coincidences were found, but were reset after each determination.

It is customary to use the term independent only for measurements that are at the same time neither dependent nor conditioned.

How can density be determined by a direct measurement? If so determined would the mass, density, and volume of an object be independent, dependent, or conditioned?

Harmony and Disagreement of Repeated Measurements.—If the same object is measured several times in succession the measurements will in general differ from one another, but the differences will tend to be small under two different circumstances: (*a*) if the quantity is of a sharply defined character, and (*b*) if the method of measurement is coarse or crude. Thus, if a length of woolen cloth were compared with an accurate millimetre scale it would probably be found difficult to measure it the same twice in succes-

sion; but if the same length of steel rail were clamped on rigid supports and measured at a constant temperature it might be hard to obtain two measurements that would differ. The length of one object would be a poorly defined quantity; that of the other, a sharply defined quantity. Although sharpness of definition of the quantity to be investigated means that repeated measurements will tend to harmonize closely, it should be carefully noted that accuracy of the methods or means of measurement has just the opposite result; it is the rough methods of measurement that make repeated determinations identical with one another, and the refined methods that show discrepancies. Two similar 1-lb. weights may appear to have precisely the same mass on a rough balance and yet differ when weighed on a more carefully constructed one. If the heavier weight should then be filed down just enough to make it equal to the lighter one a test with a delicate chemical balance might show not only that the two were still unequal but even that one of them alone would not weigh the same amount twice in succession.

The general statement can be made, then, that an accurately defined quantity or a coarse method of measurement will result in a series of determinations being harmonious or identical, while a poorly defined quantity or an accurate method of measurement will cause the results to disagree or diverge.

It is a general truth that, no matter how sharply defined a quantity may be, the use of the most precise methods will result in successive equally careful measurements of it differing perceptibly from one another, although the differences may be very small.

Thus, the most accurate possible determinations of the length of a national prototype metre would undoubtedly differ by several tenths of a micron.

It necessarily follows that the true magnitude of a measured quantity is always unknown; and the various approximations to it need to be summarized, for actual use, by their average or by some other representative value.

Errors of Measurement.—The *error* of a measurement is the amount by which it differs from the true value of the quantity which is measured. If the true value is always unknown the error must likewise be unknown. Such errors, however, can be discussed theoretically, and in this way much can be learned about the best manner of dealing with them.

Classification of Errors.—Errors are classified as *constant* and *accidental*, and what are known as *mistakes* really belong in a separate class by themselves. *Constant errors* affect all the measurements of a series in the same manner or in the same direction. *Accidental errors* make one measurement a trifle too large and another too small, but do not tend to bias the average result. *Mistakes* are occasional errors that are due to a lack of mental alertness on the part of the observer.

ERRORS

CONSTANT:

- THEORETICAL: usually calculable, as the faulty length of a linear scale, due to its expansion from heat.
- INSTRUMENTAL: due to faulty graduation or adjustment of an instrument.
- PERSONAL: some persons have a constant tendency to estimate the instant of an occurrence a little too early; others a little too late.

ACCIDENTAL:

- INSTRUMENTAL: due to varying external influences, "play" of moving parts, inconstant sensitiveness, etc.
- PHYSIOLOGICAL: the senses of sight, touch, etc., have a limit of sensitiveness, and this limit is not always the same.
- PSYCHOLOGICAL: very likely the deductions about the outside world that result from the effect of sense-impressions on the mind do not correspond to the latter so closely as to be absolutely free from irregular variations when dealing with minute quantities.

MISTAKES:

- MANIPULATIVE: doing the wrong thing.
- OBSERVATIONAL: observing the wrong thing.
- NUMERICAL: recording the wrong numbers; it is especially important to guard against focussing the attention so closely on the minute part of a measurement (*e. g.*, the estimation of tenths of the smallest scale-division) that a mistake is made in recording the figures that express the larger part of it.

Accidental and Constant Errors.—The difference between constant errors and accidental errors can be easily understood from the analogy of shots fired at a target. In the diagram the average position or centre of clustering has a constant tendency downward and to the right of the bull's-eye, while the individual shots have accidental tendencies which carry them to one side of this average position as much as

to the other. Any single shot can be considered to have a total error which is the vector sum or geometrical combination of its own accidental error plus the constant error of the whole group. Physical measure-

ments are target shots in which the centre of clustering can be found but in which the position of the bull's-eye is unknown. Constant errors can be avoided or reduced to a minimum by the help of theoretical knowledge (effect of gravity in deflecting the shots downward), and by changing observers, methods, and conditions (repeating the shots when the wind blows to the left instead of to the right), and above all by judgment, experience, care, and alert hunting for all possible sources of error. Accidental errors are much more easily investigated, the chief problems from the standpoint of physical science being the determination of the centre of the cluster and the measurement of the degree of scattering which takes place around it.

State what sources of error were present when you performed each of the experiments mentioned in the following list. Mark them in such a way as to show which ones gave rise to constant errors, and which ones caused accidental errors. Were there any mistakes?

The measurement of your span.

The measurement of an irregular area, first method.

The measurement of the density of an irregular solid.

The measurement of the sine of an angle that has a given tangent.

The experimental determination of π.

The calculation of e^{-x^2} for different values of x.

The use of the formula $1/(1 + \delta) = 1 - \delta$.

The "black-thread" determination.

Errors and Variations.—The theory of accidental errors runs closely parallel with the theory of variation in natural objects, so that a statistical investigation

of the properties of several objects of the same kind, or *variates* as they are technically called, furnishes a good illustration of many of the facts which could otherwise be obtained only by the more tedious study of the variations in repeated measurements of the same object.

Measurement of Variates.—Examine the vernier caliper, and if it has an extra scale that reads backwards, or one scale for internal measurements and another for external ones decide which scale reads the internal distance between the jaws of the caliper, and which point marks its zero when the jaws are closed. See that you understand the vernier and have no trouble in reading it at any setting.

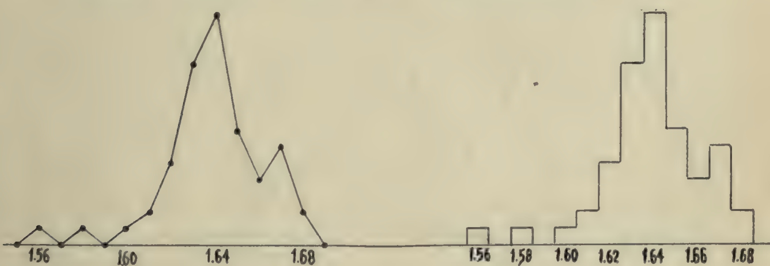
cm	*frequency*
1.56	/
1.58	/
1.59	
1.60	/
1.61	//
1.62	/////
1.63	///// ///// /
1.64	///// ///// ////
1.65	///// //
1.66	///
1.67	///// /
1.68	//

Measure the length of 100 seeds or other variates of the same kind. Make a few preliminary measurements to find their general range and then make a table like the above, recording each different length and the number of times that it occurs. If an extreme value is found later it may be noted anywhere at the beginning or end of the table, as shown here.

Plot a graphic diagram in which the measurements in centimetres are laid off along the x-axis and the frequency of each length is represented by the height of a corresponding ordinate. Connect the points of the diagram by a broken line and notice that the curve is highest in the middle and slopes downward, more or less uniformly, toward the ends.

XI. STATISTICAL METHODS

Frequency Distributions.—A tabulation of a set of measurements that shows how many times each observed value occurs is said to give the "frequency distribution" of the measurements, and a graphic diagram such as was drawn in the preceding exercise is known as a frequency polygon. A slightly different method of drawing essentially the same diagram is by using a length on the x-axis, instead of a point, to represent what is called the "class-interval" of the



frequency distribution, or least difference between successive values of the measurement. Rectangles are then constructed on the short base lines so that their height represents the frequency of the corresponding measurement. This type of figure is called a "histogram," and will be seen to be practically the same as the frequency polygon.

Class Interval.—If the class interval is made very small and the measurements are very numerous, both forms of diagram can be considered as losing their abrupt changes until they merge into two identical

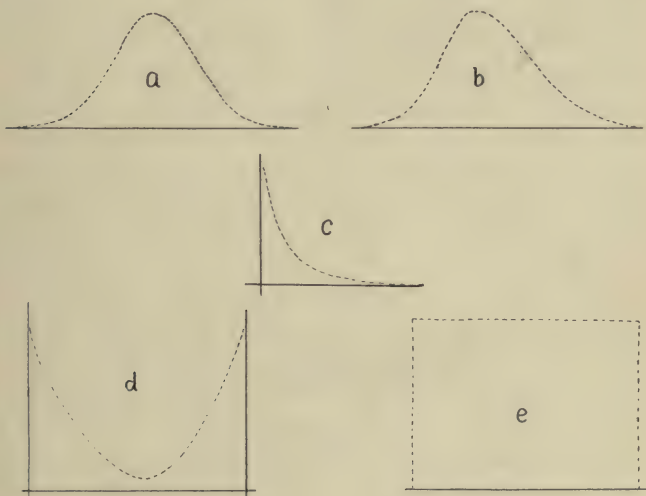
curved lines. If the class interval is small, however, without a sufficiently great number of measurements the frequency of successive values will tend to give a series of ordinates like 0, 1, 0, 0, 1, 0, 1, 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0, 1, 0, and the frequency diagram will no longer have its characteristic "mound-like" shape with the ordinates high in the middle of the diagram and dwindling toward each end.

If your frequency polygon tends to the nondescript type seen when the class interval is too small the measurements of the hundred variates should be regrouped. Make a new table in which the values from 1.55 to 1.65 are all called 1.6, those from 1.65 to 1.75 are considered as 1.7, etc. If several measurements have been recorded as 1.65 put about half of them in the class 1.6 and the other half in the class 1.7. From this table plot both the frequency-polygon and the histogram that illustrate it graphically.

Types of Frequency Distribution.—Frequency distributions are classified, according to the general shape of the graphic diagram, as (*a*) symmetrical, (*b*) modererately asymmetrical, (*c*) very asymmetrical or J-shaped, (*d*) bilocular or U-shaped, (*e*) rectangular.

The symmetrical type (*a*) is seen in physical measurements: the average x-value is the most frequent; measurements that are a given amount above or below the average are less frequent than it but of equal frequency with each other; and extremely divergent values do not occur. The asymmetrical type (*b*) is similar except that the most frequent value does not lie in the middle of the distribution and on one side of it the frequency falls away more rapidly than

on the other. Moderate asymmetry is common in all statistical data. An example of the J-shaped type (*c*) may be seen in the distribution of wealth among any population; the frequency of individuals with little wealth is very great, and with increasing wealth the number of cases falls off until it reaches a vanishing point. The curious U-shaped type (*d*) is seen where



there are tendencies toward both extremes, or the centre is a position of unstable equilibrium; and the rectangular type (*e*) occurs in purely mathematical cases, such as the actual error of a tabulated logarithm, which is never more than ± 0.5 of the unit in the last decimal place and has all intermediate values with equal frequency.

The Probability Curve.—The measurement of 100 seeds will probably show a moderate degree of asymmetry, since such objects do not grow beyond a definite size but do fall short of it in many cases. Physical measurements, however, tend to be above the average just as often as they are below it, and so give the symmetrical form of frequency distribution. As the measurements are made more and more numerous the frequency polygon approaches more and more closely to the form of the so-called *probability curve*, $y = e^{-x^2}$, which was calculated in the lesson on logarithms and plotted in the lesson on graphic representation. In some cases it will appear drawn out relatively flat and in others will be very high and narrow, but the curve is the same in all cases, except that the scales of x-values and y-values are condensed or spread out to different degrees.

Carry out the binomial expansion that is given below at least as far as $n = 15$, adding each two terms in order to obtain the term below and between them. Lay off a series of equal intervals on the x-axis and erect a series of ordinates proportional to the successive terms of the last polynomial, in order. A smooth curve through the tops of the ordinates will give a very good approximation to the probability curve.

$$
\begin{aligned}
(1+1)^0 &= 1 \\
(1+1)^1 &= 1 + 1 \\
(1+1)^2 &= 1 + 2 + 1 \\
(1+1)^3 &= 1 + 3 + 3 + 1 \\
(1+1)^4 &= 1 + 4 + 6 + 4 + 1 \\
(1+1)^5 &= 1 + 5 + 10 + 10 + 5 + 1 \\
(1+1)^n &=
\end{aligned}
$$

Representative Magnitudes.—For most scientific work, the statement of a whole frequency distribution or of each one of a long series of measurements would be a cumbrous process and reading such statements would be a tedious task. Accordingly it is usual to summarize such a set of values by stating some representative value, such as the average. In special cases such representative magnitudes as the geometrical mean, $\sqrt[n]{(a_1 a_2 \ldots . a_n)}$, or the harmonic mean, $n/(1/a_1 + 1/a_2 + \ldots . + 1/a_n)$, or the quadratic mean, $\sqrt{[(a_1^2 + a_2^2 + \ldots + a_n^2)/n]}$, have been employed, but the most usual one is the arithmetical mean or *average*, $(a_1 + a_2 + \ldots + a_n)/n$. The *median*, which is a_n if $a_1, a_2 \ldots a_{2n-1}$ are arranged in order of size, is frequently useful as a representative value, as is also the *mode* or *modal value*, which is simply the value that occurs with the greatest frequency.

The Average.—The average is obtained by adding together a set of values and dividing the sum by the number of values. Thus the average of the five values 3, 3, 4, 5, 10, is one-fifth of their sum, or 5. The average of the measurements given in the table at the end of the preceding lesson is $(2 \times 1.68 + 6 \times 1.67 + 3 \times 1.66 \ldots)/(2 + 6 + 3 \ldots)$, or 8686/53 or 1.639.

The Median.—The median is obtained by choosing such a value that half of the other values exceed it and half are below it. If the numbers are arranged in numerical order in a column the number that is half way down the column is the median. Otherwise it may be found by crossing off the largest number and the smallest, and repeating the process until only one is left. If two numbers are left, as will be the case if

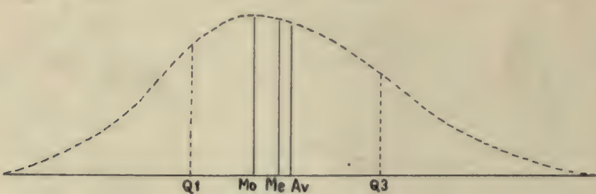
there are an even number of measurements, the number half-way between them can be taken as the median, but in physical or statistical data it usually happens that the two remaining numbers are the same.

Find the median of 3, 3, 4, 5, 10. What is the median of 7.4, 6.8, 7.3, 7.3, 7.2, 7.1, 7.2? *Ans:* 7.2. Find the median of the measurements that were averaged in the preceding paragraph.

The Mode.—The mode is the most frequent value. Thus in the set of values 3, 3, 4, 5, 10 the mode is the number that occurs twice, namely 3. In the illustration of the measurement of variates the mode is 1.64, the length that was found most often.

Choice of Means.—If a frequency distribution is of the symmetrical type the average, mode, and median will all be the same.

Where there is a *moderate degree of asymmetry* the median will come nearer to the mode than the average does, as is shown in the following diagram. The



mode is easily seen to be the most probable value. If a seed is taken at random from the set whose measurements are given above it is more likely to measure 1.64 cm. than any other amount. The mode has one decided disadvantage, however, in that it cannot

always be chosen from a given frequency distribution. For example the mode cannot be obtained from a series like 3, 4, 4, 5, 6, 6, 7, 8, except by assuming that the values follow the probability law, and fitting a probability curve to them as accurately as possible by a "black thread" method; the position of the top of this theoretical curve can then be determined. Where there is a long series of measurements the mode is sometimes calculated from the formula, indicated on the above diagram, that $me - mo = 2\ (av - me)$, or that the median lies $\frac{1}{3}$ of the way from the average to the mode.

The median, like the mode, is easy to determine. It can be used in two classes of cases where the average cannot be determined. One is in case measurements have been tabulated with indefinite terminal classes, *e. g.*, "less than 10 mm., 7 cases; 10 to 12 mm., 4 cases; 12 to 14, 5; 14 to 16, 3; above 16, 2 cases." Here the median is the class "between 10 and 12," say 11 mm., and the mode is probably the class "12 to 14," say 13 mm., but the average cannot be found on account of nine of the numerical values not being stated; the best that could be done would be to guess that the average was a little less than the median. The other case is where quantities can be arranged in numerical order but are difficult to measure individually. Thus it may be difficult or impossible to gauge the scholarship of a student in accurate numerical terms, but if a group of students can be arranged in order of scholarship the median can be determined without difficulty. Furthermore the median has an advantage over most other representative magnitudes in that it is not affected

by inverting the unit of measurement. The median of a number of prices will be the same whether they are given as cents per dozen or as dozens per dollar. Of a group of different velocities the same one will be picked out by choosing the median whether the numerical values are expressed in miles per hour or in minutes per mile. It is easy to see that this will not be the case if the average is used. The median, however, is not quite as good a representative value as the average, in case they are different, for a series of measurements that have all been made with equal care. Curiously enough, however, the more extensive a series of measurements the more likely it is to show that the individual measurements are not equally trustworthy but may be grouped in different classes according to their relative scattering. It is in such cases that the median is a much better representative figure than the average, for the average is influenced by an unduly large or small measurement just in proportion to the aberration of the latter, while as long as a measurement is above the median it makes no difference how far above it may be, its effect is no greater than that of any other single value (Compare the average and the median of 3, 3, 4, 5, 10, with those of 3, 3, 4, 5, 30.)

Find the average, the mode, and the median of your measurements of 100 variates. What is the ratio of mode minus median to median minus average?

Deviations.—Of almost as much importance as finding the best representative value for a series of measurements is the determination of how closely they cluster around it or how widely they scatter from it. This will help to furnish information in regard to

the accuracy of the measurements, and hence also in regard to the accuracy of the instruments and methods employed in making them; it will also be useful in comparing and combining determinations made at different times or by different observers. The arithmetical difference between any single measurement and the average, or other representative magnitude, is known as the "deviation" of that measurement. It is not the same as the error of the measurement, for the error may have a constant component that affects all of the measurements equally; but it may be considered as the accidental error or accidental component of the total error. Deviations give no positive indication of any constant errors that may be present.

If each one of a series of independent measurements is as trustworthy as any other it can be shown mathematically that their best representative value is their arithmetical mean, or average; and accordingly the average is the figure that is almost invariably used.

Copy the two following columns of figures, find the average of each set, and write after each measurement its deviation from the average of the figures in the same column, marking it with a minus sign if the value is less than the average, and with a plus sign if in excess of the average.

27.35	27.36
27.34	27.38
27.34	27.35
27.33	27.37
27.34	27.32
27.34	27.30
27.33	27.31
27.34	27.30
27.35	27.35
27.34	27.36

Although both averages are the same it is evident that the first set of measurements must have been made by a more trustworthy instrument, observer, or method than the second. If the averages had not been of the same

value the first one would undoubtedly have been entitled to more confidence, other things being equal, than the second.

Add each column of deviations and verify the fact that the algebraical sum of the deviations from the average is always zero. If their sum is zero their average will also be zero, so that when an "average deviation" is spoken of the term means not the average of the algebraical deviations but the average of the positive arithmetical values of all of them.

Average by Symmetry.—The foregoing property suggests an easy method of finding the average in simple cases: If a number can be so chosen that the individual measurements are symmetrically grouped around it the sum of the positive deviations will equal the sum of the negative deviations and the number will be the required average.

What is the average of 115 and 119? *Ans:* 117, because it makes the sum of the deviations (+2 and −2) equal to zero.

Find the average of 3, 6, 9, 12, 15 by the method of symmetry.

What is the average of 16, 18, 20, 22? Of 14 and 17? Of 12, 14, 17, 19? Of 126.8 and 127.4? Of 121 and 141? Of 161 and 191? Of 198, 199, 203? Of 8, $10\frac{1}{2}$, $11\frac{1}{2}$? What is the value, to the nearest whole number, of the average of 117, 116, 117? (Suggestion: Is the average above or below 116.5?) What is the exact average of 17, 18, 18, 19, 19, 20?

Average by Partition.—Before leaving the subject of the average it should be noted that there is no need of adding the entire numerical values if they contain a

part in common. Thus in either of the columns of figures on page 97 the average is obtained by writing the first three figures, which are common to all values, and the average of the last figure.

Quartiles.—Just as the middle number of a series arranged in ascending or descending order of magnitude is called the median, or half-way value, so the middle one of the numbers that lie below the median is called the *lower quartile*, or quarter-way value; and the median of the numbers that are greater than the median is known as the *upper quartile*, or three-quarter-way value. The quartile abscissæ are laid off on the base line of the diagram, on page 94, at q_1 and q_3. It should be carefully noticed that their ordinates, together with the median ordinate, divide the whole area of the frequency diagram into four equal parts. If there is any difficulty in understanding this turn to the histogram on page 89, and notice that the area indicates the total number of measurements. If half of them lie below the median, by definition, and half above there can be no trouble in realizing that the median is the abscissa whose ordinate bisects the area of the figure. Of course the same thing is not true of the average, except when average and median happen to have the same value.

What are the quartiles of the series 3, 4, 4, 5, 6, 8, 11?

Find the median and quartile values of the set of numbers 28, 29, 31, 36, 31, 30, 35, 33, 32, 36, 29, 28, 32.

Semi-Interquartile Range.—The numerical distance between the lower quartile and the upper quartile is called the *interquartile range*. In the series 7, 8, 10, 13, 19, 22, 27, the quartiles are 8, and 22, and their

difference, 14, is the range between quartiles, or interquartile range. Half of this, or 7, is called the *semi-interquartile range;* notice that it is the average of the two distances from median to quartile. It is sometimes used as a measure of the scattering or clustering of a set of observations, and its theoretical importance will be seen a little later.

Find the semi-interquartile range of each of the columns of figures on page 97. For the first column: if the five smaller values are considered as lying below the median, that is, if the median is considered to occupy no numbers in the middle of the column the lower quartile will be 27.34; if only four values are considered as lying below the median, that is, if the median is considered to include two numbers in the middle of the column the lower quartile will be 27.335; as the median is actually to be considered as one single number it will be satisfactory to say that the lower quartile is half-way between these two values, say 27.338. Similarly, the upper quartile may be taken as 27.342. The second column should be treated in thesame manner.

XII. DEVIATION AND DISPERSION

Apparatus.—Vernier caliper; variates; slide rule.

Characteristic Deviations.—Just as the use of an average or other representative magnitude makes it unnecessary to state the separate measurements from which it is derived, so a statement of all the deviations from the average can be replaced by a single *characteristic deviation.* A set of measurements can then be summarized by two numbers, and it is customary

to write such a result in the form $a \pm d$, where the first number gives the average value of the quantity measured and the second expresses the limiting distances above and below the average which include half of the measurements or which mark off in some other way an amount of deviation which is characteristic for the set of measurements.

Total Range.—The simplest way of summarizing the deviations of a series of measurements is by stating their *total range*, or the algebraical difference between the smallest and largest. Obviously the extreme range of the measurements themselves will also give the same result. Thus, in the two columns of measurements that have been considered above the total range is .02 for the first and .08 for the second. Unfortunately, this is also the worst method of obtaining a characteristic deviation, for the total range depends upon only two of the measurements of the series; and those two are the least satisfactory ones, because repetitions of the series of measurements would undoubtedly show a considerable fluctuation in the largest and smallest values, while the most closely clustered values would hardly be changed at all.

Average Deviation.—A better index of the amount of scattering is given by the *average deviation.* This is the average of the positive arithmetical values of the deviations. The true algebraical average of the deviations cannot be used if they have been calculated from the average measurement because, as has been shown, its value is always zero.

The average of the positive values of the deviations is always smallest when the deviations have been cal-

culated from the median measurement, and it is in connection with the median that it is generally used.

Find the average and the median of the numbers 3, 3, 4, 5, 10. What is the average deviation from the median? What is it from the average?

Standard Deviation.—The typical deviation that is most frequently used for statistical purposes is the *standard deviation.* This is the square root of the quotient of the sum of the squares of the deviations from the average divided by one less than the number of statistical values.* If there are n quantities whose deviations from their average are respectively d_1, d_2, d_3, . . . d_n, the standard deviation of those quantities is the value of

$$\sqrt{\frac{d_1^2 + d_2^2 + d_3^2 + \ldots + d_n^2}{n-1}}.$$

The standard deviation is a special case of the *mean-square deviation*, being in fact the mean-square deviation from the average, or approximately the quadratic mean of the deviations from the average. It is also sometimes called the *mean deviation* and the *mean-square* deviation, but as the term *mean deviation* is also used for what we have defined as the *average deviation* it is much better to avoid the use of the word *mean* altogether in connection with a deviation.

Dispersion.—The measure of deviation which is used for physical measurements in almost all cases is that

* To be strictly accurate, the standard deviation of the statistician is $\sqrt{\Sigma d^2} / \sqrt{n}$. The reasons for using $n-1$ will be found in the mathematical works; here it need only be noticed that the denominator is diminished because the numerator is smaller than the sum of the squares of the true errors (page 124). Of course, if n is fairly large, the distinction is unnecessary.

particular form of characteristic deviation which is called the *dispersion.* It is approximately two-thirds of the standard deviation, or, more exactly

$$\pm .6745 \sqrt{\frac{d_1^2 + d_2^2 + \ldots + d_n^2.}{n-1}}$$

Significance of the Dispersion.—An important characteristic of the dispersion is that for a series of measurements which is extensive enough to follow the law of the probability curve this typical value can be shown mathematically to express exactly the same limits above and below the average as are given with respect to the median by the semi-interquartile range. If measurements follow the law of the probability curve the median and the average will coincide; and the upper and lower quartiles include just half of the total number of measurements within the space between them, as was seen in connection with the curve on page 67. In other words, the right-hand half of the histogram will be bisected by the ordinate corresponding to the positive value of the dispersion, and the corresponding abscissa will be the median value of all the positive deviations. Since the left-hand half of the curve is likewise bisected by the negative value of the deviation, and the graphic diagram is symmetrical with respect to the y-axis, it follows that the dispersion is the median of the absolute values of all the deviations, or that *any single deviation is as likely to be less than the dispersion as it is to be greater.* The dispersion, with its plus-and-minus sign marks off a small distance around the average, and it is just as likely as not that any single measurement chosen at random will be within these limits.

Advantage of the Dispersion.—Where the number of measurements in a series is relatively small, say not greater than 10, the dispersion obtained by calculation and the semi-interquartile range obtained by picking out the quartile values will not usually be of the same value, on account of the measurements not being sufficiently numerous to follow the laws of probability very closely. Even in this case, however, the value given by the formula is preferable to half the difference between the quartiles because the former is obtained from all the measurements of the series and the latter from only two. Where there are as many as ten measurements of a physical magnitude it is usually found that the two values will not differ by more than 15 or 20 per cent., and for rough preliminary measurements the semi-interquartile range is almost as satisfactory as the dispersion and can be obtained much more readily.

v	d	d^2
1.82 cm	33	1089
1.85	63	3969
1.85	63	3969
1.78	7	49
1.82	33	1089
1.84	53	2809
1.78	7	49
1.62	167	27889
1.81	23	529
1.70	87	7569
17.87 (10		49010 . . . $\Sigma(d^2)$
1.787 . . av.		

$$\begin{aligned} \log 49010 &= 4.6903 \\ \log 9 &= \ \ .9542 \\ \hline & 2)3.7361 \\ & \ \ 1.8680 \\ \log .6745 &= 1.8290 \\ \hline & 1.6970 = \log \mathrm{D} \\ \mathrm{D} &= 49.78 \end{aligned}$$

Calculation of the Dispersion. — Measure ten variates taken at random from the 100 that were measured before, and tabulate them in a column headed v. Find the average and write the

deviations without decimal points in an adjacent column d. In a third column write the respective values of d^2, using the table of squares on page 141 to obtain them. Then find the sum of the squares, divide it by $n - 1$, 9 in this case, and multiply the square root of the quotient by .6745. The product will be the dispersion.

If the numbers in column d are expressed in thousandths of a centimetre the dispersion will also come out in thousandths; thus, the value of D shown in the illustration means .04978 cm.

Rule for Accuracy of the Average.—In determining how many figures of the average are to be retained as significant it is best to follow the rule that *at least half of the deviations should be greater than* 3. In the illustrative example it will be noticed that keeping three decimal places in the average has made more than half of the deviations greater than 30. In such a case the average could obviously be rounded off to two decimal places, 1.79, and at least half the deviations would still be greater than 3. This will be found to simplify the calculation and the result will not be essentially different. In the above example the result would have come out 4.982 (hundredths of course, this time, not thousandths), or .04982 cm, which is seen to agree with the previous result to three significant figures, and two significant figures are all that is usually wanted for the value of a dispersion.

Use of the Table of Dispersions.—The root extraction and long multiplication and division should, of course, never be done by the tedious arithmetical process. Even the logarithmic process used in the

illustration can be avoided as follows: $49010 \div 9 = 5446$; the square root of this will be somewhat more than 70; $\frac{2}{3}$ of 70 is about 48; in the column headed $(n \pm \frac{1}{2})^2/.6745^2$ of the table on page 141 find two numbers between which 5446 lies and read the corresponding number in column n. It is immediately seen to be 50, which agrees with the previously obtained 49.78 as far as the two significant figures which are all that is required. As any arrangement of significant figures has two square roots (see page 60) a place for 5446 would also be found opposite 16 in the table, but the rough preliminary check-calculation showed that the answer should be about 48, so there could be no doubt that the required answer is 50 rather than 16.

Dispersions with the Slide Rule.—In all following work in physics dispersions are to be calculated either with the table, as explained, or with the slide rule, which makes the process even easier. A special line is marked at 6745 on the C scale so that $.6745\sqrt{a/b}$ can be obtained with a single setting as soon as the sum of the squares of the deviations is obtained. Difficulty with the double square root is best avoided by setting the end of the C scale to the approximate value of the radical as obtained by a rough mental calculation; a very slight movement of the slide is then all that is needed to make an exact setting.

Sigma-notation.—The capital letter sigma (Σ) of the Greek alphabet is often used by mathematicians, prefixed to an algebraical term, to denote the sum of all such terms; thus the expression for the average $(a_1 + a_2 + a_3 \ldots + a_n)/n$ is abbreviated to the equivalent form $(\Sigma a)/n$. In the same way the formula

$$.6745\sqrt{\frac{d_1^2 + d_2^2 + \ldots + d_n^2}{n-1}}$$

is easier to write in the form

$$.6745 \sqrt{\Sigma(d^2)/(n-1)},$$

and the use of $\Sigma(d^2)$ will have been noticed previously in the example of the calculation of a dispersion.

Rewrite the formulæ for the harmonic mean and the quadratic mean (page 93), using the sigma-notation. Write the formula for $(x_1 + x_2)^2$, using the same notation.

If five measurements of the quantity x give 3, 3, 4, 5, 10, what is the value of Σx? If the average is 5 what is the value of Σd? Of $\Sigma(d^2)$? Of $\Sigma(x^2)$?

Dispersion of an Average.—The average length of 10 variates has already been calculated. If 10 more were measured their average would be somewhere nearly the same as the first one. If a considerable number of such averages had been determined it would be a simple matter to determine their dispersion, and the result would naturally be a smaller number than the dispersion of any set of individual measurements, since an average is a more trustworthy figure than a single determination. In fact it can be shown mathematically that if ten equally good measurements are averaged the single measurements will show a variation which is greater than the variation of such averages in the proportion of $\sqrt{10}$ to 1, and similarly that the dispersion of averages will be $1/\sqrt{n}$ as great as the dispersion of individual measurements if the latter are averaged in groups of n. This means that it is not necessary to calculate several averages in order to find their dispersion,

for it can be determined from a knowledge of the number of measurements that go to make up a single average and from the dispersion of these individual measurements. Thus the *dispersion of the average*, as it is called, of nine measurements is at once seen to be one-third as large as the dispersion for single measurements, since the square root of nine is three.

The formula for the dispersion of an average is easily written, for if

$$D_1 = .6745\sqrt{\frac{\Sigma\,(d^2)}{n-1}},$$

then

$$D_{av} = .6745\ \sqrt{\frac{\Sigma\,(d^2)}{n(n-1)}},$$

the second formula being $1/\sqrt{n}$ as large as the first.

The Statement of a Measurement.—Tabulate the first eleven of the twelve measurements of the wooden block made with the card-board model of a vernier caliper. Pick out their semi-interquartile range, to be used as a rough value of the dispersion of the individual measurements, and divide it by the square root of n. Calculate the dispersion of the average and compare it with the approximation just found, employing the usual rule for comparison: "divide the difference by the greater number." Write the thickness of the block in the form $av. \pm D_{av}$, the customary method of stating a measurement of any kind. Divide D_1 for your measurement of 10 seeds by $\sqrt{10}$ in order to obtain D_{av} for them, and state their measurement in the same form.

Make eleven more measurements of the wooden block, this time with the vernier caliper that gives

tenths of a millimetre, and treat them in the same way as the previous eleven. What can you learn from the results obtained with the two different pieces of apparatus?

Relative Dispersion.—It has already been shown that in order to see how serious an error really is it should be divided by the true magnitude of the quantity measured. In the same way it is often found that the dispersion itself does not give as much information as the *proportional dispersion,* or *relative dispersion,* or *fractional dispersion,* as it is also called; the ratio of dispersion to representative magnitude.

In the two measurements just stated, the length of a seed and the thickness of the wooden block, divide the dispersion of the average by the average itself in order to obtain the *relative dispersion of the average.* Express this either as a decimal or as a percentage. For which set of measurements should you expect it to be the smaller? Why? Does the same relation hold true for your dispersions of the averages as expressed in centimetres? Explain why.

XIII. THE WEIGHTING OF OBSERVATIONS

Apparatus.—Platform balance and clamp; set of weights; vernier caliper; aluminum block; over-flow can and catch-bucket for measuring displaced water; string and two spreading rods; fine silk thread; slide rule.

Necessity of Weights for Observations.—A representative value is often wanted for measurements which are not all equally trustworthy. The accepted values for such constants as the maximum density of water,

the mechanical equivalent of heat, the length of the true ohm of mercury, the velocity of light *in vacuo*, have all been derived from measurements by different observers at various times, and in general by different apparatus and methods. Any of these varying factors will produce varying results, and one determination can be accepted with more confidence than another and so will be entitled to greater "weight" when it necessary to decide upon a representative value.

Density by Different Methods.—An example of the effect of different methods on the determination of a physical magnitude may be given by the measurement of the density of a metal block. If the mass is known this can be accomplished by mensuration, by measuring displacement, or by a measurement of buoyant force. According to the Principle of Archimedes the apparent loss of weight of a body immersed in a fluid is the same as the weight of an equal volume of the fluid. If the volume of a metal block is v, its weight w, and its apparent weight in water w', the density can be found as the ratio of the weight, w, to the loss of weight, $w - w'$, supposing that the density of water is unity; or it can be determined as the ratio of the weight, w, to the volume of water that is actually displaced on immersion, say v'; or the block can be measured with a caliper and the density calculated as m/v.

Clamp the platform balance to the cross-bar above the table in such a way that an object can be weighed by suspending it under the bar with strings attached to a spreading rod on one scale-pan of the balance. Use the other spreading rod as a counterpoise, and make a careful allowance for the fact that they may

not exactly balance. Attach the aluminum block to the string by a fine thread long enough to allow it to hang within the empty overflow can, and weigh it as accurately as possible. Fill the overflow can with water, closing the spout with the finger-tip; place it in position where the aluminum block is to hang, with the catch-bucket under the spout; allow the excess of water to run out of the overflow can; then weigh the catch-bucket with its contained water, and replace it in position. Lower the aluminum block carefully into the overflow can and weigh it while submerged; then weigh the catch-bucket again in order to find out how much water was displaced.

Find the density of the aluminum block (*a*) by comparing its weight with the weight of water actually displaced; (*b*) from the two values w and w'; (*c*) by measuring the block with the vernier caliper, computing its volume as closely as possible, and applying the formula for density, $d = m/v$. Report your results for comparison with those of the other members of the class.

Weights for Repeated Values.—The simplest case of weighting different observations is when separate numerical values have each been obtained a definite number of times. Suppose, for example, that the density of a block of aluminum has been determined both as 2.6 and as 2.7, the smaller value having been found on four occasions while the larger value was obtained only once in the total of five measurements. The best representative figure from these data certainly would not be the number 2.65, half way between 2.6 and 2.7, but ought to be a number situated four times as far

from the least frequent measurement, 2.7, as from the most frequent one, 2.6; in other words, it should be the number 2.62. Moreover, this is easily seen to be the same result as would be obtained by taking the average of the five individual measurements. The rule in such a case is obviously to give each numerical value a weight proportional to the number of times of its occurrence.

Find the *weighted average* of the values of a measured length if it was found to be 2.345 cm in each of six trials, 2.350 cm in twelve trials, and 2.355 in nine trials. Suggestion: calculate the value of $2 \times 2.345 + 3 \times 2.355 + 4 \times 2.350$.

The Weighted Average.—The weighted average is found in any case by considering that certain values have been obtained more frequently than others. In the case just discussed this was a fact, in other cases it is only a supposition made to fit the known or estimated intrinsic value of the observations.

If a difficult measurement had been made by an experienced student and found to be 0.35, while the same experiment gave the value 0.41 when performed by a beginner, it might be decided somewhat arbitrarily to give the first number twice the weight of the second. The process of finding the weighted average, $(2 \times 0.35 + 1 \times 0.41)/3$, would then be equivalent to supposing that the better measurement had been obtained on two occasions but the poorer one only once. If a measurement of some quantity had been found to be 1.36 when made under unfavorable circumstances, and 1.41 when made under circumstances that were more favorable to experimentation it might be considered

best to assign the respective weights of 1 and 1.5 to the two values. The weighted average would then be $(2 \times 1.36 + 3 \times 1.41) \div 5$, or 1.39, a figure which will be seen to be nearer to the better value than to the poorer one in exactly the ratio of 1 to 1.5.

Arbitrarily Assigned Weights.—The objectionable feature of such an arbitrary assignment of weights is very obvious. The relative weights depend too much upon the judgment of the individual computer; furthermore, it is often difficult to avoid being influenced by the fact that certain determinations vary more or less widely from the expected value, instead of keeping one's judgment focussed on the quality of the experimental work.

Which do you consider the better method of determining density, by buoyancy or by displacement? Choose what you consider the best ratio for their relative accuracies and find the corresponding weighted average, but be careful not to give extra weight to either measurement on account of its coming close to the third determination made by calculating the volume obtained by mensuration.

Weight and Dispersion.—Determinations of any carefully measured magnitude are usually stated in the form of an average and its dispersion. If the influence of constant errors can be neglected it can be shown mathematically that the best value for the measurement is obtained by *weighting each determination in inverse proportion to the square of its dispersion.* Thus, if one determination has a dispersion of .0040 and another has a dispersion of .012 the former should be given nine times as much weight as the latter.

This can be expressed in a general formula by saying that the weighted average of

$$a_1 \pm d_1, \ a_2 \pm d_2, \ a_3 \pm d_3, \ \ldots .$$

is

$$\frac{a_1/d_1^2 + a_2/d_2^2 + a_3/d_3^2 + \ldots}{1/d_1^2 + 1/d_2^2 + 1/d_3^2 + \ldots}$$

or

$$w.\, av. = \Sigma(a/d^2)/\Sigma(1/d^2),$$

but it is much better to learn the principle involved than to memorize the formula.

Limitations of $\mathbf{w = k/d^2}$.—Attention should again be directed to the fact that weighting according to dispersions takes no account of the fact that constant errors may be present in the given data. The dispersion summarizes only the accidental errors, and if the constant errors are greater than these the weighted average is no better than the simple arithmetical average.

Tabulate the determinations, made by the various members of the class, of the density of aluminum as found by the effect of buoyancy. Calculate the typical value in the form $a \pm d$.

Find in the same way the average and dispersion of the density as determined by displacement.

Calculate the weighted average of these two data.

There are several sources of constant error in each of the two above methods of determining density. State at least four that are common to both methods, and at least one that influences one form of experiment but not the other.

Exception to the Rule.—The method of weighting observations in inverse proportion to their dispersions is

used for separate and independent data whose relative accuracy is assumed to be shown by their dispersions. Where two or more series of observations, however, are known to have been made with equally reliable apparatus, methods, and observers they should be weighted merely according to the number of measurements which each comprises, notwithstanding that their dispersions might indicate a very different result. To do otherwise would be to repudiate the principle of the average, which depends upon the fact that all observations are supposed to be equally trustworthy. On the other hand, when different observations are known to be unequally trustworthy, even if they occur in the same series, weight may be given to the fact that some are closely clustered about an apparent central position while others diverge erratically.

Which is to be preferred, the average or the median, for a determination, like the one just made, of density by buoyancy? Why? (Refer back to Lesson XI.)

XIV. CRITERIA OF REJECTION

Apparatus.—Slide rule.

Observational Honesty.—When successive redeterminations of a quantity have been made in the course of an experimental investigation it is to be supposed that they have all been made with an equal degree of care.

It is important to remember than an observation should never be rejected simply because it is not in satisfactory agreement with the other determinations of the series. If the experimenter realizes that one of

his measurements was made under some kind of a handicap or under such conditions that a faulty result would be likely it is permissible to cross out the corresponding value in his notes and to omit it in the final consideration of the data, but there must be some definite and satisfactory reason for discarding it other than the fact of its divergence from the expected value. The temptation, often felt by the beginner, to omit or "re-determine" a discordant result may be very perceptible, but absolute integrity of observation should be cultivated to such a point that the experimenter is habitually able to feel a certain disinterestedness in the outcome of a measurement after he has first taken pains to ensure its being as trustworthy as possible.

Importance of Criteria.—Even with all care to make successive measurements equally accurate it often happens that one or more of them show unduly large deviations from the average. In order to prevent these values from having an abnormal influence on the representative value certain rules have been formulated to determine whether they shall be retained or discarded, for if an observer merely used his own judgment in deciding the question the result would depend too much upon his own individuality and temperament, and different observers would obtain different results from data identically the same, just as in the case of arbitrarily assigned weights. In fact, the rejection of a measurement is nothing more nor less than giving it a weight equal to zero.

Chauvenet's Criterion.—One of the easiest to understand of the various devices for testing doubtful observations is known as *Chauvenet's criterion of rejection*,

according to which rejectability is determined as a function of deviation, dispersion, and number of measurements.

Draw a graphic diagram from the table. Draw the ordinates $x = 10$ and $x = 50$ from the base line up to the curve. The result will be the right-hand half of a normal frequency polygon, the x-values corresponding to deviations and the y-values to the frequency of their occurrence. Remember that the total area of such a curve corresponds to the number of deviations; in the same way the area between curve and base line which is bounded by any two ordinates represents the number of observations whose numerical deviations lie between those two limits. As the dispersion is the same as the median deviation it is evident that the ordinate which bisects the area (the ordinate $x = 10$, for the scales used in this diagram) must have its abscissa numerically equal to the dispersion.

Suppose another ordinate is so drawn as to include nine-tenths of the area between the y-axis and itself, and leave only one-tenth of the area beyond it to the right, then the corresponding abscissa would similarly have a value that would be exceeded by only one-tenth of the total number of deviations, and if any deviation were chosen at random there would be only one chance in ten that it would be larger than the corresponding x-value. It can be proved

x	y
0	50.0
1	49.9
2	49.6
3	49.0
4	48.2
5	47.2
6	46.0
7	44.6
8	43.3
9	41.6
10	39.9
12	36.0
14	32.1
16	28.0
18	24.0
20	20.2
22	16.6
24	13.8
26	10.8
28	8.4
30	6.5
32	4.8
34	3.6
36	2.6
38	1.8
40	1.2
42	0.9
44	0.7
46	0.5
48	0.3
50	0.2
52	0.1
54	0.1
56	0.0
58	0.0
60	0.0

mathematically that in order to include nine-tenths of the area the ordinate must be drawn 2.44 times as far to the right of the y-axis as the line which bisects the area and corresponds to the dispersion.

On your diagram draw the ordinate that includes nine-tenths of the area and make sure that its abscissa fulfills the condition stated above. If the total number of measurements were ten how many would most probably be represented by the area to the right of the ordinate? How many if the number of measurements were 50? How many if the number were 4? How many if 6? The last two questions should be answered to the nearest whole number.

Since the ordinate for $x = 2.44\ d$ includes nine-tenths of the area and the limit $2.44 \times$ *dispersion* includes nine-tenths of the deviations it might be said theoretically and rather figuratively that if there were only five measurements in a certain series the number of measurements whose deviations were greater than this limit would most probably be just *one-half.* In other words the limit would be just on such a border line that if it was decreased we should expect it to exclude one measurement rather than no measurements, and if it was increased we should expect it to exclude no measurements rather than one measurement.

It follows, then, that *no* one of a series of five measurements theoretically ought to have a deviation of more than 2.44 times the dispersion. This being the case it is only natural to consider that one is justified in discarding any one of the measurements of a series of five if its deviation does exceed this limit. Chauvenet's criterion is simply an extension of this statement

to other values of n as well as 5. The column, l, of the following table shows the limiting value of x for which the area of the curve $y = e^{-x^2}$ is $1 - 1/2n$, this value being expressed in terms of the dispersion or probable error.

n	l	$\log l$	n	l	$\log l$	n	l	$\log l$	n	l	$\log l$
1	1.00	000	11	2.97	473	21	3.35	525	32	3.59	554
2	1.71	233	12	3.02	480	22	3.38	529	34	3.62	556
3	2.05	312	13	3.07	487	23	3.40	532	36	3.65	561
4	2.27	356	14	3.11	493	24	3.43	535	38	3.68	566
5	2.44	387	15	3.15	498	25	3.45	538	40	3.70	570
6	2.57	410	16	3.19	504	26	3.47	540	49	3.81	581
7	2.67	427	17	3.22	508	27	3.49	543	64	3.95	597
8	2.76	441	18	3.26	513	28	3.51	546	81	4.06	608
9	2.84	453	19	3.29	517	29	3.53	549	100	4.16	619
10	2.91	464	20	3.32	521	30	3.55	551	676	5.00	699

Chauvenet's criterion.

It should be carefully kept in mind when considering any criterion of rejection that we are interested in the individual measurements, and, accordingly, the dispersion to which the criterion applies is the dispersion of the individual measurements, not the dispersion of the average. Chauvenet's criterion is then the test of whether any deviation is greater than l times the dispersion of the individual values of a series of n measurements.

The Probable Error.—It will be noticed that the dispersion has also been called the "probable error." By this time the student ought to be thoroughly aware of the fact that the dispersion is not an error at all, but a deviation. If he also realizes that deviations within its limits are no more probable than improbable

there can be no objection to his using the term that is always employed by physicists in speaking of this characteristic deviation. In this book the term dispersion has been used in order to avoid repeatedly informing the student that it is an error and repeatedly suggesting that there is something very probable about it. It will hereafter be spoken of as the probable error, and of course it will be understood that it is used in two forms, the probable error of the individual measurements and the probable error of the average.

In an experimental determination of the specific heat of lead shot the following values were obtained by a class of students. Test them by Chauvenet's criterion to determine whether any measurement falls outside of the theoretical limits, but if two or more such values are found reject only the most divergent one, find a new average for those that remain, and apply the criterion to them in turn. Repeat the process, if necessary, until no more values can be discarded, and then state the best value obtainable from the figures, with its "probable error."

.022
.0309
.031
.032
.0347
.035
.036
.038
.045
.049

Graphic Approximation to Chauvenet's Criterion.—Where a graphic diagram is to be used for only a single series of numbers instead of for sets of values of two varying quantities it is advisable to use a horizontal scale and lay off the individual measurements as small dots or circles unless they are sufficiently numerous to allow a good histogram to be drawn.

The following figures are the experimental values of the slope of the first "black-thread" diagram as obtained by a class of students: .60, .57, .64, .53, .48,

.59, .59, .61. Make a graphic diagram of these values, mark the median and quartiles, and lay off the semi-interquartile range to the right and left of the median as many times as is indicated by Chauvenet's criterion.

In this way a rough application of the criterion can be made graphically and the long calculation can be avoided. Determine from the diagram whether any value should be rejected and then verify the result by the usual form of calculation.

Use the graphic method for applying Chauvenet's criterion to the following set of barometer readings:

29.986	inches
29.982	"
29.990	"
29.984	"
29.984	"
29.980	"
29.986	"
29.977	"
29.984	"
29.982	"
29.986	"
29.988	"
29.984	"

What advantage has the arithmetical method over the graphic method?

Write down, in your own words, just what it is that is represented by (*a*) the probable error of a single measurement, $.6745\sqrt{\Sigma d^2/(n - 1}$, and by (*b*) the probable error of the average, $.6745\sqrt{\Sigma d^2/n(n - 1)}$.

Irregularities of Small Groups.—The probable error, or "dispersion," cannot be considered as having much meaning in cases where the total number of measurements is less than ten, and even with ten measurements it should be treated with a certain amount of caution. A number of values less than ten will hardly ever give a histogram of their frequency-

distribution which is recognizably similar to the graph of $y = e^{-x^2}$, the curve which all unbiassed measurements will be found to follow if they are sufficiently numerous.

Justification of the Criterion.—Likewise, it is hardly worth while to use a criterion of rejection for less than ten measurements. The example given above with only five is intended merely for an illustration of the method of using the criterion, and the still smaller values in the table are only of theoretical importance. Chauvenet's criterion is not to be considered as showing that any one measurement is a mistake, but merely as indicating that a very large deviation is such a rarity that it would have an unduly large influence upon the average if it were allowed to remain along with the other values of a very limited series of measurements.

Wright's Criterion.—Another criterion of rejection, which is sometimes employed, is that of Wright. According to this the arbitrary rejection of a single measurement may be considered if its deviation is more than five times the probable error.

Turn back to the graphic diagram of the table on page 117, and notice how small a part of the area of the curve lies to the right of the ordinate, $x = 50$, representing five times the probable error. Turn to the table of values for Chauvenet's criterion and note how many measurements would need to be made before "half a measurement" would be likely to diverge from the average five times as far as the probable error.

In the measurements to which you have already applied Chauvenet's criterion how many would have

been rejected if Wright's criterion had been used instead? What are the relative advantages of the two criteria?

Other limiting values, which give practically the same result as Wright's criterion, are *four times the average deviation*, and *three times the standard deviation.*

Comparison of Characteristic Deviations.—Turn back to your notes on the use of logarithms and find the graphic diagram of $y = e^{-x^2}$. Mark off the following values on the base line, $p = .4769$, $a = .5642$, and $s = .7071$. These represent respectively the probable error, the average deviation, and the standard deviation, and are roughly proportional to $10 : 12 : 15$; a better approximation may be found with the aid of the slide rule. Draw the corresponding ordinates, and notice that the last one meets the curve at the point of inflection, that is, at the point where it is momentarily straight as it changes from convex upward to convex downward.

XV. LEAST SQUARES AND VARIOUS ERRORS

Apparatus.—Clock, chronometer, or time circuit, giving audible seconds; watch with second-hand; slide rule.

The Average as a Least-Square Magnitude.—The mathematical *principle of least squares* is that when measurements are equally trustworthy their best representative value is that for which the sum of the squares of the deviations has the lowest numerical value. It is upon this principle that the use of the

average is based, for it is easy to show that the sum of the squares of the deviations of any particular set of numbers (try 3, 3, 4, 5, 10) will be greater when measured from some other value (try the median) than when measured from the average.

Least Squares for Conditioned Measurements.— If we are dealing with two conditioned measurements, as in the case of the x- and y-values of the black-thread experiment, the principle of least squares shows that the line which expresses the *condition* or gives the law of relationship between the two variables must be so placed that the sum of the squares of the distances from it to all of the experimental points shall have the smallest possible value.

The x and y of any one of the points cannot in general be substituted in the black-thread equation, $y = a + bx$, but $a + bx - y$ will have some small positive or negative value instead of being equal to zero. If the various points are considered to have the definite positions (x_1, y_1), (x_2, y_2), (x_3, y_3), etc., it will be found that none of these sets of values will satisfy the equation $y = a + bx$ or $a + bx - y = 0$, but will give such a result as $a + bx_1 - y_1 = d_1$, where d is some small quantity whose exact value need not be determined; similarly, the other points will give $a + bx_2 - y_2 = d_2$, $a + bx_3 - y_3 = d_3$, etc., and according to the principle of least squares the sum $d_1^2 + d_2^2 + d_3^2 + \ldots$, must be as small as possible.* This means that the sum of the left-hand members of the equations, or $\Sigma(a + bx_n - y_n)^2$ must have its minimum value, and it can be shown

* It can be shown mathematically that the distance from (x_1y_1) to $y = a + bx$ is proportional to $a + bx_1 - y_1$.

by processes of pure mathematics that this will be the case if

$$a = \frac{\Sigma\,(x)\,\Sigma\,(xy) - \Sigma\,(y)\,\Sigma\,(x^2)}{(\Sigma x)^2 - n\,\Sigma\,(x^2)}$$

and

$$b = \frac{\Sigma\,(x)\,\Sigma\,(y) - n\Sigma\,(xy)}{(\Sigma x)^2 - n\,\Sigma\,(x^2)}.$$

By similar processes a, b, and c could be found for the equation of the parabola $y = a + bx + cx^2$, or the appropriate coefficients for curves having even more complicated equations, but the processes of computation becomes so tedious that it is better to replace the variables, as explained in the lesson on graphic analysis, by others that will conform to the straight line law.

Tabulate the values of x and y for the black-thread experiment on page 70, arranging them as shown in the following table, but writing the proper numerical values in the spaces marked Σx, Σy, $\Sigma(xy)$, and $\Sigma(x^2)$. Then calculate the values of a and b from the formulæ, arranging the work neatly and being careful to avoid using the wrong algebraical signs or confusing $\Sigma(x^2)$ with $(\Sigma x)^2$. Keep only three significant figures in the final results.

x	y	xy	x^2
1	9.8	9.8	1
2	8.5	17.0	4
3	8.0	24.0	9
4	7.2	28.8	16
5	6.7	33.5	25
6	6.5	39.0	36
7	6.2	43.4	49
8	5.5	44.0	64
9	5.0	45.0	81
10	4.1	41.0	100
11	3.9	42.9	121
12	3.2	38.4	144
13	2.3	29.9	169
Σx	Σy	$\Sigma(xy)$	$\Sigma(x^2)$

Write the equation representing the best position of the black thread in the form $y = a + bx$, and then in the form $x/a + y/b = 1$. Compare the calculated values of the intercepts with

your experimental values obtained in the lesson on graphic analysis.

Probable Errors of Indirect Measurements.—All the probable errors which have been considered previously have been probable errors of direct measurements. Suppose an indirect measurement is to be obtained from certain direct measurements by the use of an appropriate formula. If one set of values of the direct measurements were substituted in the formula in order to calculate one value of the indirect measurement, and then another set in order to calculate a second value, and so on, it would be an easy matter to find the probable error of the resultant set of indirect measurements but the repeated calculations would be much more laborious and time-consuming than the process of substituting the average of each direct measurement. Furthermore, some of the values needed in the formula may be predetermined constants, such as those mentioned on page 109, which are given only in the form of a representative magnitude and its probable error. The question then arises as to the way in which the probable error of the indirect measurement is influenced by the size of the probable errors of the direct measurements on which it is based.

The simplest case is when the indirect measurement is merely the sum of the two independent direct measurements. Let the direct measurements be

$$a_1 \pm p_1 \qquad \text{and} \qquad a_2 \pm p_2,$$

when stated in the form of average and probable error, and let the indirect measurement with its probable error be

$$\mathrm{A} \pm \mathrm{P},$$

so that $A = a_1 + a_2$; then it can be proved that

$$P^2 = p_1^2 + p_2^2.$$

Similarly if $A = a_1 - a_2$ it is also true that P^2 is equal to $p_1^2 + p_2^2$, not $p_1^2 - p_2^2$.

If

$$A = c_1 a_1,$$

where c_1 is some constant not subject to error, it can be shown that

$$P^2 = c_1^2 p_1^2 \text{ or } P = c_1 p_1.$$

In general, if

$$\mathbf{A = c_1 a_1 \pm c_2 a_2 \pm c_3 a_3 \pm \ldots}$$

then

$$\mathbf{P^2 = c_1^2 p_1^2 + c_2^2 p_2^2 + c_3^2 p_3^2 + \ldots}$$

where p_n is the probable error of the average a_n.

This formula can be used to find the probable error of an algebraical expression when the probable error of each of its terms is known.

If two independent measurements are multiplied together the probable error of the product will follow the law expressed in the following equation.

If

$$A = a_1 a_2$$

then

$$P^2 = p_1^2 a_2^2 + a_1^2 p_2^2.$$

Likewise if

$$A = a_1 a_2 a_3$$

then

$$P^2 = (p_1 a_2 a_3)^2 + (a_1 p_2 a_3)^2 + (a_1 a_2 p_3)^2,$$

and similarly for any number of factors; but it is more satisfactory in practice to make use of the *relative* probable error (relative dispersion, page 109) as in the following form, which is easily deducible from the form just given.

If

$$A = a_1\, a_2\, a_3 \ldots .$$

then

$$(P/A)^2 = (p_1/a_1)^2 + (p_2/a_2)^2 + (p_3/a_3)^2 + \ldots .$$

If

$$A = a_1{}^n$$

then

$$(P/A)^2 = (n^2\, p_1{}^2/a_1{}^2) \text{ or } P/A = np_1/a_1;$$

and, likewise, if

$$A = c_1\, a_1{}^n$$

then

$$P/A = np_1/a_1,$$

the constant not appearing in the formula if the relative probable error is used.

In general, whether the exponents are positive, negative, or fractional, if

$$\mathbf{A = c_1 a_1{}^m a_2{}^n a_3{}^r \ldots .}$$

then

$$\mathbf{(P/A)^2 = (mp_1/a_1)^2 + (np_2/a_2)^2 + (rp_3/a_3)^2 + \ldots}$$

This formula can be used to find the probable error of any single term of an algebraical expression when the probable errors of its factors are known.

A bowl whose interior is an exact segment of a sphere is found to have a depth of 25.00 ± .02 centimetres and a diameter across the top of 50.00 ± .30 centimetres. Find its capacity from the formula for the volume of a spherical segment, $v = \pi hr^2/2 + \pi h^3/6$, where h is the height or depth of the segment and r is the radius of its circular base; find the probable

error of the capacity by applying the second general equation to each term of the formula and then using the first equation to determine the final result. Notice the relative probable error of the radius, r, is the same as that of the diameter, d. Arrange the calculation systematically in order to avoid numerical mistakes, and if there is any trouble in making the substitution write out each step of the process; for example:

$a_1 = 25.$	$P^2/A^2 = 3^2(.02)^2/25^2$	$\log 25 = 1.3979$
$p_1 = .02$	$P^2 = 3^2(.02)^2\pi^2 25^6/25^2 6^2$	$\log \pi^2 \quad \underline{0.9943}$
$m = 3$	$= \pi^2(.01)^2 25^4$	$\bar{4}.0000$
$c_1 = \pi/6$		5.5916
$A = \pi h^3/6$		2.5859
		$\log P =$

Notice that an indirect probable error which depends upon the measurement of two or more different quantities always assumes the form $\sqrt{x^2 + y^2}$, and consequently will be more decidedly diminished by reducing the larger of the two independent probable errors than by attempting to improve the more accurate measurement. Show that $\sqrt{5^2 + 2^2}$ is reduced by 41% if the 5 is changed to 2.5, but only by 5% if the 2 is changed to 1.

Systematic Errors.—It has been shown that errors may be either accidental or constant. There is another class of errors, often included under the term constant errors, in which the error is not actually constant, nor does it vary according to the law of probability. This is the class of *systematic errors*, or errors that undergo a more or less regular change during the course of making a set of measurements. They may be subdivided into *progressive errors*, which show a steady

increase (or decrease) from one determination to the next, and *periodic errors* which increase for a number of measurements, then decrease, and then repeat the previous cycle or period. Where systematic errors are absent a comparison of any measurement of a series with the preceding one will tend to show an increase in the numerical value about as often as a decrease; a fact that can easily be tested by writing between each two successive values a plus sign, a minus sign, or a zero, according as the second value is greater than, less than, or equal to, the first, and then comparing the number of the plus signs with that of the minus signs. Where progressive errors have been greater than accidental errors there may be all plus signs or all minus signs as the result of applying the test; and if the systematic errors are periodic there will be alternate groups of plus signs and minus signs.

In an experiment in which water in a reservoir was drawn up into a tube by suction and successive readings of its height were made values having the following decimals were obtained in order: .76, .74, .70, .62, .63, .61, .55, .56, .51, .50, .44, .44, .39, .40, .35, .35. Are the results probably affected by progressive errors, or periodic errors, or neither? Why?

Stand where you can hear the clock beat seconds and read the time indicated by your watch. Every seven seconds as indicated by the clock read the seconds and estimated tenths of a second from the watch and state the result to another student, who will take down the values in his note-book. After three or four minutes change places with him and note down the time as he reads it off. Every seven-second interval should have

its time by the watch noted, for a full period of seven minutes.

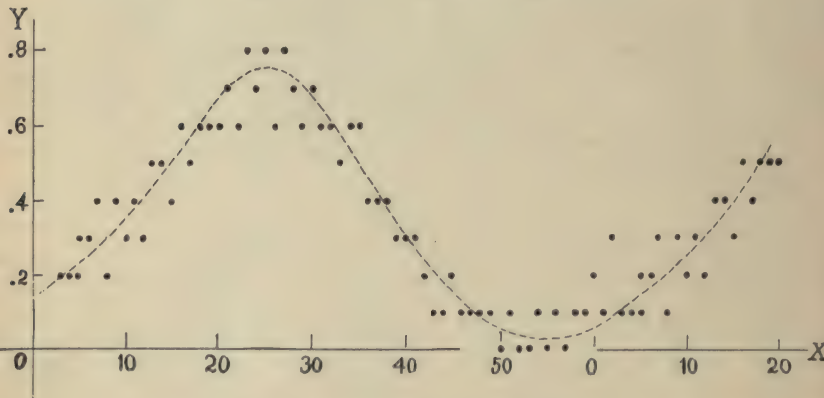
See that you have the complete table of sixty values in your own note-book, and mark the observed tenths of a second with a plus sign where they increase from one observation to the next and with a minus sign where they decrease. With most watches it will be found that the second hand is not pivoted in the exact centre of the graduated circle and the periodic error will be shown very distinctly.

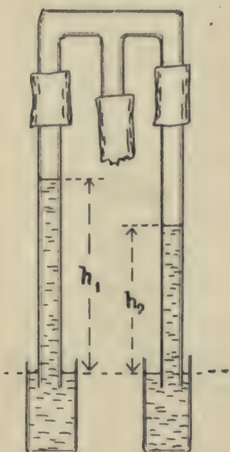
hr min sec		*hr min sec*		*hr min sec*	
4:37:65.2	+	4:40:25.8	−	4:42:45.2	−
38:12.3	+	32.6	−	52.0	+
19.6	O	39.3	−	59.1	+
26.6	−	46.1	−	43:06.3	+
33.5	−	53.0	+	13.5	+
40.3	−	60.2	+	20.6	+
47.1	O	41:07.4	+	27.8	−
54.1	O	14.6	+	34.6	−
39:01.1	+	21.7	O	41.3	−
08.2	+	28.7	−	48.1	−
15.4	+	35.6	−	55.0	+
22.6	O	42.2	−	44:02.3	+
29.6	−	49.1	O	09.4	+
36.4	−	56.1	+	16.6	+
43.1	−	42:03.2	+	23.8	−
50.0	O	10.3	+	30.7	−
57.0	+	12.5	+	37.4	−
64.2	+	24.7	−	44.1	O
40:11.4	+	31.6	−	51.1	O
18.6	+	38.4	−	58.1	+
				4:45:05.3	

Draw a graphic diagram in which the abscissæ represent the integral part of the number of seconds in the table, and the ordinates represent the corresponding tenths of a second. Draw a smooth curve to eliminate accidental errors in the determination of time.

Summarize the result by stating "the pivot of the

second hand of the watch is displaced toward the figure... of the dial by an amount equal to the length of... seconds' divisions on the graduated circle."





Explain how the periodic error can be eliminated in case such a watch is used for determining intervals of time.

The list of figures given on page 130 was obtained from a determination of specific gravity by Hare's method. If the lower ends of two upright tubes dip into two separate reservoirs while their upper ends are both joined to a third tube from which the air can be partially exhausted it can easily be

proved that the heights to which the fluids are raised will be inversely proportional to their densities; so that if a fluid whose density is unity is raised to a height h_1, and a heavier fluid to a lesser height h_2, the density or specific gravity of the latter must be h_1/h_2. The complete list of determinations of height included readings of both columns of liquid; they were made at approximately equal intervals of time, and in the order in which they are given in the table.

If the density is calculated by dividing 75.76 by 73.06 it is evident that the resulting figure will be too large, for the height of the water had fallen somewhat below 75.76 when the reading of the salt solution, 73.06, was taken; and if 75.74 is divided by 73.06 the result will be too small, for the salt solution did not stay at 73.06 while the reading 75.74 was being taken. Obviously the average of 75.76 and 75.74 must be divided by 73.06, or 75.74 must be divided by the average of 73.06 and 73.04, or some other combination used in which the average height of one column of liquid must have occurred at the same time as the average height of the other. This method of eliminating progressive errors is used in the process of weighing with a delicate bal-

pure water	*salt solution*
75.76	
	73.06
75.74	
	73.04
75.70	
	73.00
	72.98
	.95
75.62	
	.94
.63	
.61	
	.89
	.88
.55	
.56	
	.84
	.84
.51	
.50	
	.79
	.78
.44	
.44	
	.75
	.70
.39	
.40	
	.77
	.77
.35	
.35	

ance and in many other processes of physical measurement.

What set of values near the end of the table can be used in the same way? Make five different calculations of density from successive parts of the table and see whether they show any evidence of progressive error.

Constant Errors.—It has already been stated that constant errors are more troublesome than accidental errors and that the latter give very little aid in determining the former. It is not the target (page 86) that is found from individual measurements but only the centre of clustering, and characteristic deviations show only how close determinations come to one another, not how close they come to the truth.

Some constant errors are easily corrected with the aid of theoretical considerations; others may be very difficult to eliminate. Unfortunately there is no infallible rule for detecting them, and each experimental problem has its own special sources of error. The two beam-arms of a balance may be unequal, so that all weighings are proportionately erroneous; the end of a metre-stick may be worn, so that every setting of the zero-point is inaccurate; the neutral tint of litmus may be faultily judged, so that a chemical determination is biassed. Consider such a simple process as the determination of atmospheric pressure with a mercurial barometer. The vacuum at the top is never perfect and there is often capillary action, both making the reading too low. If the barometer and its attached scale do not hang vertically every apparent reading will be too high. The scale itself is too long or too short except at a single temperature, and the mercury

may have its accepted standard density only at a different temperature from the one that it has when the observation is made. Even if its density is standard the height of a column that will give a definite pressure will depend upon the strength of gravitational attraction and this varies with the latitude and altitude of the instrument. If an aneroid barometer is to be used instead of a mercurial one its mechanism introduces still more sources of error.

It is evident that the amount of constant error can be varied by changing observers, apparatus, methods, and times of observation; and the more radically different the sets of conditions are made the better, in all probability, will be the mutual neutralization of constant errors when the weighted average is taken. In practice, the values for most of the constants of nature have been obtained under such varying conditions. Atomic weights are obtained from various inter-relations of chemical compounds obtained from different sources and by different methods. The surface tension of water may be measured by the hanging drop method, by the capillary wave method, by the vibrating jet method, etc. The size of the molecules of a gas may be calculated from the rate at which heat is conducted through them, from the "covolume constant," b, of Van der Waal's equation, from experimental determinations of the viscosity of the gas, from measurements of the maximum density obtainable by cooling and liquefying or solidifying it, etc. If various determinations agree closely in spite of the employment of essentially different methods it becomes more probable that constant errors have been satisfactorily removed,

but it can never be certain that all of these methods have not some common source of error which would be eliminated only by using some entirely different method. Constant watchfulness, as stated on page 87, and the exercise of good judgment are of the greatest importance in guarding against constant errors. In the student's future laboratory work various "sources of error" that have been found by previous experimenters will be explicitly stated. Many of them will be sources of constant error, and both his natural ability and his progress in learning will be tested by his treatment of them.

TABLES

EXPLANATORY NOTE

The use of the table of squares will be self-evident. Notice that the square of a number between 100 and 110, say of 107, consists of five figures which are, in order, 1, $2n$, n^2, or 1, 14, 49. The square of any number between 100 and 200 can be found by the same process, "carrying" mentally. Thus

$$
\begin{array}{rl}
112^2 = & 1 \\
 & 24 \\
 & \;\;144 \\
\hline
 & 12544
\end{array}
\qquad\qquad
\begin{array}{rl}
173^2 = & 1 \\
 & 146 \\
 & \;\;5329 \\
\hline
 & 29929
\end{array}
$$

If either $\Sigma(d^2)/n$ or $\Sigma(d^2)/n(n-1)$ is located between two consecutive numbers in the third column, $(n \pm 1/2)/(.67449)^2$, of the same table, then the value of $.6745\sqrt{\Sigma(d^2)/n}$ or $.6745\sqrt{\Sigma(d^2)/n\,(n-1)}$, as the case may be, will be found opposite it in the first column. A very rough mental calculation will prevent taking a value which is $\sqrt{10}$ times too large or small.

The table of circular functions gives the "radian value," natural sine, cosine, tangent, and cotangent for every degree of the quadrant, also the logarithmic sine and cosine. By subtracting the two latter from each other and from zero any of the six logarithmic functions may be obtained from the table by inspection. Sines and cosines of any intermediate values can safely be obtained by interpolation, and tangents up to tan 70°. For the sine, tangent, and numerical measure of

a small angle the equations at the corners of the table should be used.

In the four-place logarithm table, as well as in the five-place table, the approximate tabular difference has been given at such frequent intervals as to make it unnecessary to subtract anything except the final digit. In the five-place table proportional parts have been omitted, as they cannot be made trustworthy in such a table on two pages. Interpolations may be calculated on paper; or, after a little practice, can easily be worked out mentally by using three-figure logarithms taken from the table, and adding them from left to right in order to calculate products. To interpolate an antilogarithm mentally, proceed as follows: Using three decimal places take out first the cologarithm of the tabular difference; add it to the logarithm of the tabular excess[1] and the antilogarithm of the sum will give the fourth and fifth figures of the required number.

When three-figure logarithms are wanted they should be taken from the five-place table. On the first page or in the first column of the second page a final 50 or 500 is simply to be dropped (as indicated by the minus sign after it); elsewhere the preceding figure is to be increased by unity.

[1] *I. e.*, the difference between the given logarithm and the next lower tabular value.

n	n^2	$\frac{(n\pm\frac{1}{2})^2}{.67449^2}$	n	n^2	$\frac{(n\pm\frac{1}{2})^2}{.67449^2}$
10	100	21762	60	3600	77819
11	121	24234	61	3721	80456
12	144	29070	62	3844	83138
13	169	34345	63	3969	85864
14	196	40061	64	4096	88633
15	225	46215	65	4225	91447
16	256	52810	66	4356	94304
17	289	59843	67	4489	97206
18	324	67317	68	4624	10015
19	361	75230	69	4761	10314
		83583			10617
20	400	92375	70	4900	10925
21	441	10161	71	5041	11237
22	484	11128	72	5184	11554
23	529	12139	73	5329	11875
24	576	13194	74	5476	12200
25	625	14293	75	5625	12530
26	676	15436	76	5776	12864
27	729	16623	77	5929	13202
28	784	17854	78	6084	13545
29	841		79	6241	
		19129			13893
30	900	20448	80	6400	14244
31	961	21811	81	6561	14600
32	1024	23218	82	6724	14961
33	1089	24668	83	6889	15326
34	1156	26163	84	7056	15695
35	1225	27702	85	7225	16069
36	1296	29284	86	7396	16447
37	1369	30911	87	7569	16829
38	1444	32581	88	7744	17216
39	1521		89	7921	
		34296			17607
40	1600	36054	90	8100	18003
41	1681	37857	91	8281	18403
42	1764	39703	92	8464	18808
43	1849	41594	93	8649	19217
44	1936	43528	94	8836	19630
45	2025	45506	95	9025	20047
46	2116	47529	96	9216	20469
47	2209	49595	97	9409	20896
48	2304	51705	98	9604	21327
49	2401		99	9801	
		53859			21762
50	2500	56057	100	10000	22201
51	2601	58299	101	10201	22645
52	2704	60585	102	10404	23094
53	2809	62915	103	10609	23547
54	2916	65289	104	10816	24004
55	3025	67707	105	11025	24465
56	3136	70169	106	11236	24932
57	3249	72675	107	11449	25402
58	3364	75225	108	11664	25877
59	3481		109	11881	
		77819			26356

SYMBOL	CONSTANT	LOGARITHM
π	3.141593	0.4971499
$\angle 1$	$180°/\pi = 57°17'45''$	
	$= 57°.29578$	1.7581226
	$= 3437'.747$	3.5362739
	$= 206264''.80625$	5.3144251
e	2.718282	0.4342945
M	0.4342945	9.6377843

RAD	DEG	TAN	SIN	LOG SIN	LOG COS	COS	COT	$1' = 10^{-4} \times$ 2.90888	
0000	0	0000	0000	$-\infty$	0	1	∞	90	$\pi/2$
0175	1	0175	0175	2419	9999	9998	5729	89	1553
0349	2	0349	0349	5428	9997	9994	2864	88	1536
0524	3	0524	0523	7188	9994	9986	1908	87	1518
0698	4	0699	0698	8436	9989	9976	1430	86	1501
0873	5	0875	0872	9403	9983	9962	1143	85	1484
1047	6	1051	1045	0192	9976	9945	9514	84	1466
1222	7	1228	1219	0859	9968	9925	8144	83	1449
1396	8	1405	1392	1436	9958	9903	7115	82	1431
1571	9	1584	1564	1943	9946	9877	6314	81	1414
1745	10	1763	1736	2397	9934	9848	5671	80	1396
1920	11	1944	1908	2806	9919	9816	5145	79	1379
2094	12	2126	2079	3179	9904	9781	4705	78	1361
2269	13	2309	2250	3521	9887	9744	4331	77	1344
2443	14	2493	2419	3837	9869	9703	4011	76	1326
2618	15	2679	2588	4130	9849	9659	3732	75	1309
2793	16	2867	2756	4403	9828	9613	3487	74	1292
2967	17	3057	2924	4659	9806	9563	3271	73	1274
3142	18	3249	3090	4900	9782	9511	3078	72	1257
3316	19	3443	3256	5126	9757	9455	2904	71	1239
3491	20	3640	3420	5341	9730	9397	2747	70	1222
3665	21	3839	3584	5543	9702	9336	2605	69	1204
3840	22	4040	3746	5736	9672	9272	2475	68	1187
4014	23	4245	3907	5919	9640	9205	2356	67	1169
4189	24	4452	4067	6093	9607	9135	2246	66	1152
4363	25	4663	4226	6259	9573	9063	2145	65	1134
4538	26	4877	4384	6418	9537	8988	2050	64	1117
4712	27	5095	4540	6570	9499	8910	1963	63	1100
4887	28	5317	4695	6716	9459	8829	1881	62	1082
5061	29	5543	4848	6856	9418	8746	1804	61	1065
5236	30	5774	5000	6990	9375	8660	1732	60	1047
5411	31	6009	5150	7118	9331	8572	1664	59	1030
5585	32	6249	5299	7242	9284	8480	1600	58	1012
5760	33	6494	5446	7361	9236	8387	1540	57	9948
5934	34	6745	5592	7476	9186	8290	1483	56	9774
6109	35	7002	5736	7586	9134	8192	1428	55	9599
6283	36	7265	5878	7692	9080	8090	1376	54	9425
6458	37	7536	6018	7795	9023	7986	1327	53	9250
6632	38	7813	6157	7893	8965	7880	1280	52	9076
6807	39	8098	6293	7989	8905	7771	1235	51	8901
6981	40	8391	6428	8081	8843	7660	1192	50	8727
7156	41	8693	6561	8169	8778	7547	1150	49	8552
7330	42	9004	6691	8255	8711	7431	1111	48	8378
7505	43	9325	6820	8338	8641	7314	1072	47	8203
7679	44	9657	6947	8418	8569	7193	1036	46	8029
7854	45	1	7071	8495	8495	7071	1000	45	7854
$1'' = 10^{-6} \times$ 4.84814		COT	COS	LOG COS	LOG SIN	SIN	TAN	DEG	RAD

	0	1	2	3	4	5	6	7	8	9	D
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	40
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	37
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	33
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	31
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	29
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	27
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	25
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	24
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	23
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	21
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	17
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	14
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	13
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	12
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8

P P

	43	42	41	40
1	4.3	4.2	4.1	4.0
2	8.6	8.4	8.2	8.0
3	12.9	12.6	12.3	12.0
4	17.2	16.8	16.4	16.0
5	21.5	21.0	20.5	20.0
6	25.8	25.2	24.6	24.0
7	30.1	29.4	28.7	28.0
8	34.4	33.6	32.8	32.0
9	38.7	37.8	36.9	36.0

	39	38	37	36
1	3.9	3.8	3.7	3.6
2	7.8	7.6	7.4	7.2
3	11.7	11.4	11.1	10.8
4	15.6	15.2	14.8	14.4
5	19.5	19.0	18.5	18.0
6	23.4	22.8	22.2	21.6
7	27.3	26.6	25.9	25.2
8	31.2	30.4	29.6	28.8
9	35.1	34.2	33.3	32.4

	35	34	33	32
1	3.5	3.4	3.3	3.2
2	7.0	6.8	6.6	6.4
3	10.5	10.2	9.9	9.6
4	14.0	13.6	13.2	12.8
5	17.5	17.0	16.5	16.0
6	21.0	20.4	19.8	19.2
7	24.5	23.8	23.1	22.4
8	28.0	27.2	26.4	25.6
9	31.5	30.6	29.7	28.8

	31	30	29	28
1	3.1	3.0	2.9	2.8
2	6.2	6.0	5.8	5.6
3	9.3	9.0	8.7	8.4
4	12.4	12.0	11.6	11.2
5	15.5	15.0	14.5	14.0
6	18.6	18.0	17.4	16.8
7	21.7	21.0	20.3	19.6
8	24.8	24.0	23.2	22.4
9	27.9	27.0	26.1	25.2

	27	26	25	24
1	2.7	2.6	2.5	2.4
2	5.4	5.2	5.0	4.8
3	8.1	7.8	7.5	7.2
4	10.8	10.4	10.0	9.6
5	13.5	13.0	12.5	12.0
6	16.2	15.6	15.0	14.4
7	18.9	18.2	17.5	16.8
8	21.6	20.8	20.0	19.2
9	24.3	23.4	22.5	21.6

	0	1	2	3	4	5	6	7	8	9	*D*
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	7
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	6
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039	4

P P

	23	22	21	20
1	2.3	2.2	2.1	2.0
2	4.6	4.4	4.2	4.0
3	6.9	6.6	6.3	6.0
4	9.2	8.8	8.4	8.0
5	11.5	11.0	10.5	10.0
6	13.8	13.2	12.6	12.0
7	16.1	15.4	14.7	14.0
8	18.4	17.6	16.8	16.0
9	20.7	19.8	18.9	18.0

	19	18	17	16
1	1.9	1.8	1.7	1.6
2	3.8	3.6	3.4	3.2
3	5.7	5.4	5.1	4.8
4	7.6	7.2	6.8	6.4
5	9.5	9.0	8.5	8.0
6	11.4	10.8	10.2	9.6
7	13.3	12.6	11.9	11.2
8	15.2	14.4	13.6	12.8
9	17.1	16.2	15.3	14.4

	15	14	13	12
1	1.5	1.4	1.3	1.2
2	3.0	2.8	2.6	2.4
3	4.5	4.2	3.9	3.6
4	6.0	5.6	5.2	4.8
5	7.5	7.0	6.5	6.0
6	9.0	8.4	7.8	7.2
7	10.5	9.8	9.1	8.4
8	12.0	11.2	10.4	9.6
9	13.5	12.6	11.7	10.8

	11	10	9	8
1	1.1	1.0	0.9	0.8
2	2.2	2.0	1.8	1.6
3	3.3	3.0	2.7	2.4
4	4.4	4.0	3.6	3.2
5	5.5	5.0	4.5	4.0
6	6.6	6.0	5.4	4.8
7	7.7	7.0	6.3	5.6
8	8.8	8.0	7.2	6.4
9	9.9	9.0	8.1	7.2

	7	6	5	4
1	0.7	0.6	0.5	0.4
2	1.4	1.2	1.0	0.8
3	2.1	1.8	1.5	1.2
4	2.8	2.4	2.0	1.6
5	3.5	3.0	2.5	2.0
6	4.2	3.6	3.0	2.4
7	4.9	4.2	3.5	2.8
8	5.6	4.8	4.0	3.2
9	6.3	5.4	4.5	3.6

	0	1		2	3	4		5	6	7		8	9	
100	00000	00043		00087	00130	00173		00217	00260	00303		00346	00389	43
101	00432	00475		00518	00561	00604		00647	00689	00732		00775	00817	43
102	00860	00903		00945	00988	01030		01072	01115	01157		01199	01242	42
103	01284	01326		01368	01410	01452		01494	01536	01578		01620	01662	41
104	01703	01745		01787	01828	01870		01912	01953	01995		02036	02078	41
105	02119	02160		02202	02243	02284		02325	02366	02407		02449	02490	41
106	02531	02572		02612	02653	02694		02735	02776	02816		02857	02898	40
107	02938	02979		03019	03060	03100		03141	03181	03222		03262	03302	40
108	03342	03383		03423	03463	03503		03543	03583	03623		03663	03703	40
109	03743	03782		03822	03862	03902		03941	03981	04021		04060	04100	39
11	04139	04532	390	04922	05308	05690	379	06070	06446	06819	370	07188	07555	363
12	07918	08279	358	08636	08991	09342	349	09691	10037	10380	341	10721	11059	335
13	11394	11727	330	12057	12385	12710	323	13033	13354	13672	316	13988	14301	312
14	14613	14922	307	15229	15534	15836	300	16137	16435	16732	294	17026	17319	290
15	17609	17898	287	18184	18469	18752	281	19033	19312	19590	276	19866	20140	272
16	20412	20683	269	20952	21219	21484	264	21748	22011	22272	259	22531	22789	256
17	23045	23300	253	23553	23805	24055	249	24304	24551	24797	245	25042	25285	242
18	25527	25768	239	26007	26245	26482	235	26717	26951	27184	232	27416	27646	229
19	27875	28103	227	28330	28556	28780	223	29003	29226	29447	220	29667	29885	218
20	30103	30320	216	30535	30750	30963	213	31175	31387	31597	209	31806	32015	207
21	32222	32428	205	32634	32838	33041	202	33244	33445	33646	200	33846	34044	198
22	34242	34439	196	34635	34830	35025	194	35218	35411	35603	191	35793	35984	189
23	36173	36361	188	36549	36736	36922	185	37107	37291	37475	183	37658	37840	181
24	38021	38202	180	38382	38561	38739	178	38917	39094	39270	175	39445	39620	174
25	39794	39967	173	40140	40312	40483	171	40654	40824	40993	169	41162	41330	167
26	41497	41664		41830	41996	42160	163	42325	42488	42651		42813	42975	161
27	43136	43297		43457	43616	43775	158	43933	44091	44248		44404	44560	156
28	44716	44871		45025	45179	45332	152	45484	45637	45788		45939	46090	150
29	46240	46389		46538	46687	46835	147	46982	47129	47276		47422	47567	145
30	47712	47857		48001	48144	48287	143	48430	48572	48714		48855	48996	140
31	49136	49276		49415	49554	49693	138	49831	49969	50106		50243	50379	136
32	50515	50651		50786	50920	51055	133	51188	51322	51455		51587	51720	131
33	51851	51983		52114	52244	52375	129	52504	52634	52763		52892	53020	128
34	53148	53275		53403	53529	53656	126	53782	53908	54033		54158	54283	124
35	54407	54531		54654	54777	54900	123	55023	55145	55267		55388	55509	121
36	55630	55751		55871	55991	56110		56229	56348	56467		56585	56703	117
37	56820	56937		57054	57171	57287		57403	57519	57634		57749	57864	114
38	57978	58092		58206	58320	58433		58546	58659	58771		58883	58995	111
39	59106	59218		59329	59439	59550		59660	59770	59879		59988	60097	109
40	60206	60314		60423	60531	60638		60746	60853	60959		61066	61172	106
41	61278	61384		61490	61595	61700		61805	61909	62014		62118	62221	104
42	62325	62428		62531	62634	62737		62839	62941	63043		63144	63246	101
43	63347	63448		63548	63649	63749		63849	63949	64048		64147	64246	99
44	64345	64444		64542	64640	64738		64836	64933	65031		65128	65225	96
45	65321	65418		65514	65610	65706		65801	65896	65992		66087	66181	95
46	66276	66370		66464	66558	66652		66745	66839	66932		67025	67117	93
47	67210	67302		67394	67486	67578		67669	67761	67852		67943	68034	90
48	68124	68215		68305	68395	68485		68574	68664	68753		68842	68931	88
49	69020	69108		69197	69285	69373		69461	69548	69636		69723	69810	87
50	69897	69984		70070	70157	70243		70329	70415	70501		70586	70672	85
	0	1		2	3	4		5	6	7		8	9	

	0	1	2	3	4	5	6	7	8	9	
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672	85
51	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517	83
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346	82
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159	81
54	73239	73320	73400	73480	73560	73640	73719	73799	73878	73957	79
55	74036	74115	74194	74273	74351	74429	74507	74586	74663	74741	78
56	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511	76
57	75587	75664	75740	75815	75891	75967	76042	76118	76193	76268	75
58	76343	76418	76492	76567	76641	76716	76790	76864	76938	77012	73
59	77085	77159	77232	77305	77379	77452	77525	77597	77670	77743	72
60	77815	77887	77960	78032	78104	78176	78247	78319	78390	78462	71
61	78533	78604	78675	78746	78817	78888	78958	79029	79099	79169	70
62	79239	79309	79379	79449	79518	79588	79657	79727	79796	79865	69
63	79934	80003	80072	80140	80209	80277	80346	80414	80482	80550	68
64	80618	80686	80754	80821	80889	80956	81023	81090	81158	81224	67
65	81291	81358	81425	81491	81558	81624	81690	81757	81823	81889	65
66	81954	82020	82086	82151	82217	82282	82347	82413	82478	82543	64
67	82607	82672	82737	82802	82866	82930	82995	83059	83123	83187	64
68	83251	83315	83378	83442	83506	83569	83632	83696	83759	83822	63
69	83885	83948	84011	84073	84136	84198	84261	84323	84386	84448	62
70	84510	84572	84634	84696	84757	84819	84880	84942	85003	85065	61
71	85126	85187	85248	85309	85370	85431	85491	85552	85612	85673	60
72	85733	85794	85854	85914	85974	86034	86094	86153	86213	86273	59
73	86332	86392	86451	86510	86570	86629	86688	86747	86806	86864	59
74	86923	86982	87040	87099	87157	87216	87274	87332	87390	87448	58
75	87506	87564	87622	87679	87737	87795	87852	87910	87967	88024	57
76	88081	88138	88195	88252	88309	88366	88423	88480	88536	88593	56
77	88649	88705	88762	88818	88874	88930	88986	89042	89098	89154	55
78	89209	89265	89321	89376	89432	89487	89542	89597	89653	89708	55
79	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255	54
80	90309	90363	90417	90472	90526	90580	90634	90687	90741	90795	54
81	90849	90902	90956	91009	91062	91116	91169	91222	91275	91328	53
82	91381	91434	91487	91540	91593	91645	91698	91751	91803	91855	53
83	91908	91960	92012	92065	92117	92169	92221	92273	92324	92376	52
84	92428	92480	92531	92583	92634	92686	92737	92788	92840	92891	51
85	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399	51
86	93450	93500	93551	93601	93651	93702	93752	93802	93852	93902	50
87	93952	94002	94052	94101	94151	94201	94250	94300	94349	94399	49
88	94448	94498	94547	94596	94645	94694	94743	94792	94841	94890	49
89	94939	94988	95036	95085	95134	95182	95231	95279	95328	95376	48
90	95424	95472	95521	95569	95617	95665	95713	95761	95809	95856	48
91	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332	47
92	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802	46
93	96848	96895	96942	96988	97035	97081	97128	97174	97220	97267	46
94	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727	45
95	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182	45
96	98227	98272	98318	98363	98408	98453	98498	98543	98588	98632	45
97	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078	45
98	99123	99167	99211	99255	99300	99344	99388	99432	99476	99520	44
99	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957	43
100	00000	00043	00087	00130	00173	00217	00260	00303	00346	00389	43
	0	1	2	3	4	5	6	7	8	9	

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NUMERICAL EXERCISES ILLUSTRATING PROBABLE ERRORS OF INDIRECT MEASUREMENTS

(pp. 127–128)

A bench has a height of 42.50 ± .03 cm above the floor, and a table is 44.35 ± .04 cm higher than the bench. Write the height of the table, with its probable error.

The table is 51.10 ± .04 cm lower than a shelf which is 137.95 ± .03 cm above the floor. How high is the table?

Notice that the same formula applies to both of the problems given above. The difference of two measurements has as large a probable error as their sum.

The diameter of a circular disc is found to be 7.98 ± .03 cm. What is its circumference?

A wall consists of 15 courses of bricks, each of which is 56.5 ± .5 mm thick, separated by 14 layers of mortar which have an average thickness of 7.5 ± 1.5 mm. Show that the probable error of the height of the wall is ± 22.3 mm, and state how much this figure would be reduced if the bricks were absolutely uniform. How much if the mortar was of uniform thickness instead?

A rectangular block measures 20.00 ± .04 cm in length, 10.00 ± .01 cm in breadth, and 5.00 ± .01 cm in thickness. What is its volume?

One edge of a cubical block is 10.00 ± .01 cm. What is its volume? What is the area of its total surface?

Does the rectangular block appear to have been measured as accurately as the cubical block? How do their volumes compare in respect to accuracy? Explain why.

Show that the combined volume of the two blocks is 2000. ± 4.2 cm^3.

Test the accuracy of the statements in the last complete sentence on page 107 by letting p denote the probable error of each of the n single measurements and finding the probable error of their average.

The formula for the volume of a cylinder is $v = \pi l r^2$. In determining this indirect measurement which of the two dimensions ought to be measured the more carefully? How much more carefully? Why?